Foundations of Computability Theory (10)

Nan Fang

Institute of Software, Chinese Academy of Sciences

Peking University

2025 Autumn Semester

Post's simple set

Theorem 10.1 (Post's simple set, 1944)

There exists a simple set A.

Proof.

We construct a coinfinite c.e. set $A = \bigcup_s A_s$ by stages such that for all e we meet the following requirement:

$$P_e: \ |W_e| = \infty \quad \Rightarrow \quad W_e \cap A \neq \emptyset. \tag{1}$$

Construction. Let $A_0 = \emptyset$. At stage $s+1$. Given A_s, choose the least $e < s$ such that

$$W_{e,s} \cap A_s = \emptyset \quad \& \quad (\exists x > 2e)[x \in W_{e,s}]. \tag{2}$$

If e exists, choose the least corresponding x and enumerate x in A. We say P_e receives attention at stage $s+1$. If there is no such e, go to stage $s+2$.

Verification. *Claim 1:* $\overline{A}$ is infinite.
Note that each P_i receives attention at most once thus enumerating at most one element into A greater than $2i$. Hence, there are at most e elements in $A \restriction (2e+1)$, $|\overline{A} \restriction (2e+1)| \geq e+1$.

Claim 2: Every requirement P_e is satisfied.
We prove by induction on e. For $e \geq 0$, by induction choose s_0 such that no $P_i, i < e$, satisfies (2) at any stage $s > s_0$. If W_e is infinite, choose $x > 2e$ satisfying (2) at some stage $s > s_0$. Hence, P_e receives attention at stage s, and $A \cap W_e \neq \emptyset$ thereafter. □

simple permitting construction

Theorem 10.2

For any noncomputable c.e. set C there is a simple set $A \leq_T C$.

Proof.

Let $\{C_s\}_{s \in \omega}$ be a computable enumeration of C. We modify the Post simple set construction and construct a coinfinite c.e. set A to satisfy for all e the following requirement which replaces the Post requirement (1).

$$P_e : \; |W_e| = \infty \quad \Rightarrow \quad [W_e \cap A \neq \emptyset \vee C \equiv_T \emptyset]. \tag{3}$$

Construction. Let $A_0 = \emptyset$.
At stage $s + 1$. Given A_s choose the least $e < s$ such that

$$W_{e,s} \cap A_s = \emptyset \quad \& \quad (\exists x > 2e)[x \in W_{e,s} \quad \& \quad C_{s+1} \restriction x + 1 \neq C_s \restriction x + 1]. \tag{4}$$

(Note that (4) is the same as in the Post simple set construction (2) except that the last clause demands that $(\exists y \leq x)[y \in C_{s+1} - C_s]$, i.e., that C permits x at stage $s + 1$.) Choose the least such e and enumerate the least corresponding x in A. If there is no such e go to stage $s + 2$.

□

proof cont.

Verification. Clearly, $\overline{A}$ is infinite by the same proof as that of Theorem 10.1. Furthermore, $A \leq_{\mathrm{T}} C$ because by the construction, for all x and s

$$\begin{aligned} C_{s+1} \upharpoonright x = C_s \upharpoonright x \quad &\Rightarrow \quad A_{s+1} \upharpoonright x = A_s \upharpoonright x, \\ \text{which implies} \quad C \upharpoonright x = C_s \upharpoonright x \quad &\Rightarrow \quad A \upharpoonright x = A_s \upharpoonright x. \end{aligned}$$

Claim: $(\forall e)[W_e \text{ infinite} \quad \Rightarrow \quad W_e \cap A \neq \emptyset]$.
If not, choose the least e which is a counterexample. Hence, W_e is infinite and $W_e \cap A = \emptyset$. Therefore, there is an increasing computable sequence of elements $x_1 < x_2 < \ldots$ with $2e < x_1$ and a corresponding computable sequence of stages $s_1 < s_2 < \ldots$ such that for all i we have $x_i \in W_e[s_i]$ and no element $y \leq x_i$ enters C at any stage $t \geq s_i$. Otherwise, (4) would allow $W_e \cap A \neq \emptyset$. Now define a computable function g_e such that $g_e = C$, contradicting the hypothesis that C is noncomputable. When x_i appears in $W_e[s_i]$, define $g_e(y) = C(y)[s_i]$ for all $y \leq x_i$. But $C \upharpoonright x_i$ does not change after s_i, and therefore $g_e = C$. Note that we have achieved $A \leq_{\mathrm{ibT}} C$.

□

Standard Permitting

Theorem 10.3 (Standard Permitting Theorem)

Let $f(x)$ be a computable function and let A and C be c.e. sets with computable enumerations $\{A_s\}_{s\in\omega}$ and $\{C_s\}_{s\in\omega}$, respectively, such that

$$A_{s+1} \upharpoonright x \neq A_s \upharpoonright x \quad \Rightarrow \quad (\exists y < f(x))[y \in C_{s+1} - C_s]. \tag{5}$$

Then $A \leq_{\mathrm{T}} C$. Indeed, A is bounded Turing reducible to C, $A \leq_{\mathrm{bT}} C$.

Proof.

The condition (5) guarantees that $m_A(x) \leq m_C(f(x))$. □

General Permitting Theorem

Theorem 10.4 (General Permitting Theorem)

Let A and C be c.e. sets with computable enumerations $\{A_s\}_{s\in\omega}$ and $\{C_s\}_{s\in\omega}$ and let $\hat{f}(x,s)$ be a computable function satisfying the following conditions:

1. $f(x) = \lim_s \hat{f}(x,s) > 0$ *exists.*
2. $0 < \hat{f}(x,s) \neq \hat{f}(x,s+1) \quad \Rightarrow \quad (\exists y < \hat{f}(x,s))[y \in C_{s+1} - C_s]$.
3. $A_{s+1} \upharpoonright x \neq A_s \upharpoonright x \quad \Rightarrow \quad (\exists y < \hat{f}(x,s))[y \in C_{s+1} - C_s]$.

Then $A \leq_{\mathrm{T}} C$.

Proof.

Now $f(x) = \lim_s \hat{f}(x,s) > 0$ by (i). Next, $f \leq_{\mathrm{T}} C$ by (ii) because if $\hat{f}(x,s) > 0$, then

$$f(x) = \hat{f}(x,s) \quad \Leftrightarrow \quad \neg(\exists y \leq \hat{f}(x,s))[y \in C - C_s].$$

Finally, $A \leq_{\mathrm{T}} C$ because $m_A(x) \leq m_C(f(x))$, which is C-computable. □

Reverse Permitting

Theorem 10.5 (Reverse Permitting)

If A and C are c.e. and $A \leq_{\mathrm{T}} C$, then there are computable enumerations $\{A_s\}_{s\in\omega}$ and $\{C_s\}_{s\in\omega}$ of A and C, and a computable function $\hat{f}(x,s)$ satisfying the conditions of the General Permitting Theorem 10.4.

Proof.

Fix A and C and e such that $A = \Phi_e^C$. Fix computable enumerations $\{\widehat{A}_t\}_{t\in\omega}$ and $\{\widehat{C}_t\}_{t\in\omega}$ of A and C. Now for every s let $r(s)$ be the least $t > s$ such that

$$(\forall x < s)[\Phi_e^{\widehat{C}}(x)[t] \downarrow = \widehat{A}(x)[t]]. \tag{6}$$

Clearly, r is nondecreasing. Now define $A_s = \widehat{A}_{r(s)}$, $C_s = \widehat{C}_{r(s)}$ and

$$\hat{f}(x,s) = \begin{cases} \max\{\varphi_e^{\widehat{C}}(z)[r(s)] : z < x\} & \text{if } x \leq s; \\ 0 & \text{if } x > s. \end{cases}$$

Fix x. By (6), for all $z < x \leq s$, $\Phi_e^{\widehat{C}}(z)[r(s)] = \widehat{A}(z)[r(s)]$. If $C_{s+1} \upharpoonright \hat{f}(x,s) = C_s \upharpoonright \hat{f}(x,s)$, i.e. , $\widehat{C}_{r(s+1)} \upharpoonright \hat{f}(x,s) = \widehat{C}_{r(s)} \upharpoonright \hat{f}(x,s)$, by definition of $\hat{f}$ and use principle, $\hat{f}(x,s+1) = \hat{f}(x,s)$ and $A_{s+1} \upharpoonright x = A_s \upharpoonright x$. Thus, (ii) and (iii) are satisfied.
Now let $\ell = \max\{\varphi_e^C(z) : z < x\}$, and let $t > x$ be a stage such that $C \upharpoonright \ell = C_t \upharpoonright \ell$. Then $\hat{f}(x,s) = \hat{f}(x,t) = \ell$ for all $s > t$. Thus, (i) is also satisfied. $\square$

effective simple sets

Definition 10.6

A simple set A is *effectively simple* if there is a computable function f such that

$$(\forall e)[W_e \subseteq \overline{A} \quad \Rightarrow \quad |W_e| \leq f(e)].$$

Note that Post's simple set is effectively simple via $f(e) = 2e + 1$, because if $W_e \subseteq \overline{A}$ then $W_e \subseteq \{0, 1, \ldots, 2e\}$.

effectively simple sets are complete

Theorem 10.7 (Martin, 1966)

If A is effectively simple then $K \leq_{\mathrm{T}} A$.

Proof.

Let $\{K_s\}_{s\in\omega}$ be a computable enumeration of K and $\theta(x)$ be a p.c. function such that $\theta(x) = (\mu s)[x \in K_s]$ if $x \in K$ and $\theta(x)$ diverges otherwise. Let A be effectively simple via f, $\{A_s\}_{s\in\omega}$ be a computable enumeration of A, and $\overline{A}_s = \{a_0^s < a_1^s < a_2^s < \dots\}$. By the Recursion Theorem with Parameters, define the computable function h by

$$W_{h(x)} = \begin{cases} \{a_0^{\theta(x)} < a_1^{\theta(x)} < \dots < a_{fh(x)}^{\theta(x)}\} & \text{if } x \in K; \\ \emptyset & \text{otherwise.} \end{cases}$$

Set $r(x) = (\mu s)[a_{fh(x)}^s = a_{fh(x)}]$. Then $r \leq_{\mathrm{T}} A$ because f and h are computable. (Here $fh(x)$ denotes $f(h(x))$.) Now if $x \in K$ and $r(x) \leq \theta(x)$, then $W_{h(x)} \subseteq \overline{A}$ and $|W_{h(x)}| = fh(x) + 1$, contrary to the hypothesis on f. Hence, $r(x) > \theta(x)$ for all $x \in K$. Therefore, $x \in K$ iff $x \in K_{r(x)}$. Hence, $K \leq_{\mathrm{T}} A$. □

DNC degrees

Definition 10.8

- A function f is *diagonally non-computable (DNC)* if $f(n) \neq \Phi_n(n)$ for all n such that $\Phi_n(n)\downarrow$.
- A Turing degree is DNC if it computes a DNC function.

Proposition 10.9

$\mathbf{0}'$ *is of DNC degree.*

Proof.

The following function f computable from $\emptyset'$ is DNC.

$$f(n) = \begin{cases} 1 \dot{-} \Phi_n(n) & \text{if } \Phi_n(n)\downarrow; \\ 0 & \text{otherwise.} \end{cases}$$

□

DNC and FPF

Definition 10.10

A function f is *fixed point free (FPF)* if $(\forall x)[W_{f(x)} \neq W_x]$.

Theorem 10.11 (Jockusch, Lerman, Soare, and Solovay)

A set A computes a DNC function iff it computes a FPF function.

Proof of "⇒".

Let θ be a partial computable function such that, for all e, if $W_e \neq \emptyset$ then $\theta(e) \in W_e$. By the s-m-n Theorem define a computable function h as follows.

$$\Phi_{h(x)}(y) = \Psi(x, y) = \begin{cases} \theta(x) & \text{if } \theta(x) \downarrow; \\ \uparrow & \text{otherwise.} \end{cases}$$

Then $\Phi_{h(e)}(h(e)) = \theta(e)$ for all e. Let $g \leq_T A$ be DNC. By the s-m-n Theorem define a computable function $\hat{f}$ such that $W_{\hat{f}(x)} = \{x\}$. Let $f = \hat{f} \circ g \circ h$. Then $f \leq_T A$ and $W_{f(e)} = \{g(h(e))\}$. Suppose that $W_e = W_{f(e)}$. Then $W_e \neq \emptyset$, so $\theta(e) \in W_e$, which implies that $\theta(e) = g(h(e))$. But g is DNC, so $g(h(e)) \neq \Phi_{h(e)}(h(e)) = \theta(e)$, which is a contradiction.

□

DNC degrees

Proof of "$\Leftarrow$".

Let $f \leq_{\mathrm{T}} A$ be a FPF function. Then $\Phi_{f(e)} \neq \Phi_e$ for all e. By the s-m-n Theorem define a computable function d as follows.

$$\Phi_{d(y)}(z) = \Psi(y,z) = \begin{cases} \Phi_{\Phi_y(y)}(z) & \text{if } \Phi_y(y)\downarrow; \\ \uparrow & \text{otherwise} \end{cases}$$

Let $g = f \circ d$. Note that $g \leq_{\mathrm{T}} A$. Suppose that $g(e) = \Phi_e(e)$. Then we have the following contradiction:

$$\Phi_{d(e)} \neq \Phi_{f(d(e))} = \Phi_{g(e)} = \Phi_{\Phi_e(e)} = \Phi_{d(e)}.$$

$\square$

Arslanov Completeness Criterion

Theorem 10.12 (Arslanov Completeness Criterion, 1981)

A c.e. set A is Turing complete iff it computes a FPF function.

Proof of "$\Rightarrow$".

Assume $A \equiv_T \emptyset' \equiv_T K_1 = \{e : W_e \neq \emptyset\}$. Define

$$W_{f(x)} = \begin{cases} \emptyset & \text{if } W_x \neq \emptyset; \\ \{0\} & \text{otherwise.} \end{cases}$$

The function f is clearly computable in $\emptyset' \equiv_T A$.
Now in case $W_x \neq \emptyset$, $W_{f(x)} = \emptyset \neq W_x$; In case $W_x = \emptyset$, $W_{f(x)} = \{0\} \neq W_x$. Thus, $W_{f(x)} \neq W_x$ for all x, f is FPF.

□

Modulus Lemma

Lemma 10.13 (Modulus Lemma)

If B is c.e. and $A \leq_{\mathrm{T}} B$, then $A = \lim_s A_s$ for a computable sequence $\{A_s\}_{s\in\omega}$ with a modulus $m(x) \leq_{\mathrm{T}} B$.

Proof.

Let $B = \bigcup_s B_s$ be c.e., $A = \Phi_e^B$ and define

$$A_s = \{x : x < s \quad \& \quad \Phi_e^B(x)\downarrow[s] = 1\},$$

$$m(x) = (\mu s)(\exists z \leq s)(\forall y \leq x)[\Phi_e^{B\restriction z}(y)\downarrow[s] \quad \& \quad B \restriction z = B_s \restriction z].$$

Now $\{A_s\}_{s\in\omega}$ is a computable sequence because the defining clause for A_s is computable. Furthermore, $m(x)$ is B-computable because the quantifiers are bounded, the first clause in the matrix defining $m(x)$ is computable, and the second clause is B-computable. Finally, $m(x)$ is a modulus because by the Use Principle the matrix of $m(x)$ holds for all $s \geq m(x)$.

□

Arslanov Completeness Criterion

Proof of "$\Leftarrow$".

Assume $(\forall x)[W_{f(x)} \neq W_x]$, where $f \leq_T A$. By the Modulus Lemma there is a computable function $\hat{f}(x,s)$ such that $f(x) = \lim_s \hat{f}(x,s)$ for every x, and the sequence $\{\lambda x.\hat{f}(x,s)\}_{s\in\omega}$ has a modulus $m \leq_T A$. Let $\{K_s\}_{s\in\omega}$ be a computable enumeration of K. Let $\theta(x) = (\mu s)[x \in K_s]$ if $x \in K$, and let $\theta(x)$ diverge otherwise. By the Recursion Theorem with Parameters define the computable function h by

$$W_{h(x)} = \begin{cases} W_{\hat{f}(h(x),\theta(x))} & \text{if } x \in K; \\ \emptyset & \text{otherwise.} \end{cases}$$

Now if $x \in K$ and $\theta(x) \geq m(h(x))$, then $\hat{f}(h(x),\theta(x)) = f(h(x))$ and $W_{f(h(x))} = W_{h(x)}$, contrary to the hypothesis on f. Hence, for all x,

$$x \in K \quad \Longleftrightarrow \quad x \in K_{m(h(x))},$$

and therefore $K \leq_T A$. $\square$

Proof of Martin's Theorem by Arslanov Completeness Criterion

Recall that D_y denote the finite set with canonical index y. By s-m-n Theorem define a function $\hat{f}$ as follows.

$$\Phi_{\hat{f}(x)}(y) = \Psi(x,y) = \begin{cases} 0 & \text{if } y \in D_x; \\ \uparrow & \text{otherwise.} \end{cases}$$

Then $W_{\hat{f}(x)} = D_x$.

Let A be effectively simple via h, and let $\overline{A} = \{a_0 < a_1 < a_2 < \ldots\}$. Let $f(x) = \hat{f}(y)$ such that $D_y = \{a_0, a_1, \ldots, a_{h(x)}\}$. Then $f \leq_{\mathrm{T}} A$ and $W_{f(x)} = \{a_0, a_1, \ldots, a_{h(x)}\} \subset \overline{A}$. Now if $W_e \nsubseteq \overline{A}$, $W_{f(e)} \neq W_e$. If $W_e \subseteq \overline{A}$, then $|W_e| \leq h(e)$ while $|W_{f(e)}| = h(e) + 1$. Thus, $W_{f(e)} \neq W_e$ as well. Therefore, f is a FPF function.