Foundations of Computability Theory (11)

Nan Fang

Institute of Software, Chinese Academy of Sciences

Peking University

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Kleene-Post Theorem

Theorem 11.1 (Kleene-Post, 1954)

There exist sets $A, B \leq_{\mathrm{T}} \emptyset'$ *such that* $A \mid_{\mathrm{T}} B$ *(i.e.,* $A \not\leq_{\mathrm{T}} B$ *and* $B \not\leq_{\mathrm{T}} A$*). Therefore,* $\emptyset <_{\mathrm{T}} A, B <_{\mathrm{T}} \emptyset'$.

We shall construct sequences of finite strings $\sigma_0 \prec \sigma_1 \prec \sigma_2 \prec \ldots$ and $\tau_0 \prec \tau_1 \prec \tau_2 \prec \ldots$ such that $A = \lim_s \sigma_s$ and $B = \lim_s \tau_s$. The construction of σ_s and τ_s at stage s will be computable in $\emptyset'$, then the sequences $\{\sigma_s\}_{s\in\omega}$ and $\{\tau_s\}_{s\in\omega}$ are $\emptyset'$-computable sequences. Therefore, $A, B \leq_{\mathrm{T}} \emptyset'$. To ensure that $A \not\leq_{\mathrm{T}} B$ and $B \not\leq_{\mathrm{T}} A$, it suffices to meet, for each e, the requirements:

$$R_e : A \neq \Phi_e^B \quad \& \quad S_e : B \neq \Phi_e^A$$

Strategy for satisfying R_e. Given σ_s, τ_s of length $\geq s$. Let $n = |\sigma_s|$. We find extensions $\sigma_{s+1} \succ \sigma_s$ and $\tau_{s+1} \succ \tau_s$ such that

$$\Phi_e^{\tau_{s+1}}(n) \downarrow \neq \sigma_{s+1}(n) \quad \text{or} \quad \Phi_e^{\rho}(n) \uparrow \text{ for all } \rho \succeq \tau_{s+1}. \tag{1}$$

Note that by (1), for any $f \succeq \sigma_{s+1}$ and $g \succeq \tau_{s+1}$ we have $f(n) \neq \Phi_e^g(n)$. Thus, requirement R_e is satisfied.

Kleene-Post Theorem

Construction. *Stage* $s = 0$. Define $\sigma_0 = \tau_0 = \emptyset$.
Stage $s + 1 = 2e + 1$. (We satisfy R_e). Given σ_s, τ_s of length $\geq s$. Let $n = |\sigma_s|$. Using a $\emptyset'$-oracle we test whether

$$(\exists t)(\exists \rho)[\rho \succ \tau_s \quad \& \quad \Phi^{\rho}_{e,t}(n) \downarrow]. \tag{2}$$

Case 1. Suppose (2) is satisfied. Find the least such pair (ρ, t). Define $\tau_{s+1} = \rho$ and $\sigma_{s+1}(n) = 1 - \Phi^{\rho}_{e,t}(n)$ so that the first disjunct of (1) holds.
Case 2. Suppose (2) fails. Then define $\sigma_{s+1} = \sigma_s\widehat{\ }0$ and $\tau_{s+1} = \tau_s\widehat{\ }0$. The second disjunct of (1) holds.
Stage $s + 1 = 2e + 2$. (We satisfy S_e). Proceed exactly as above but with the roles of σ_s and τ_s replaced by τ_s and σ_s, mutatis mutandis.

Verification. In either case of one stage $s + 1$, $|\sigma_{s+1}|, |\tau_{s+1}| \geq s + 1$. Therefore, $\lim_s \sigma_s$ and $\lim_s \tau_s$ are infinite sequences. Let $A = \lim_s \sigma_s$ and $B = \lim_s \tau_s$. By construction, $A, B \leq_{\mathrm{T}} \emptyset'$. And by (1), all requirements R_e and S_e are satisfied.

$\square$

Relativized Kleene-Post Theorem

Theorem 11.2 (Relativized Kleene-Post Theorem)

For any degree $\mathbf{c}$*, there are degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that* $\mathbf{c} < \mathbf{a}, \mathbf{b} < \mathbf{c}'$ *and* $\mathbf{a} \mid \mathbf{b}$*.*

Proof.

Fix a set $C \in \mathbf{c}$. Relativize the above proof to C, using a C'-oracle to build sets A and B such that

$$(A \oplus C) \mid_{\mathrm{T}} (B \oplus C) \quad \& \quad A, B \leq_{\mathrm{T}} C'.$$

For this, in place of (2) we use a C'-oracle to test whether

$$(\exists t)(\exists \tau_1)(\exists \tau_2)[\tau_1 \succeq \tau_s \quad \& \quad \tau_2 \prec C \quad \& \quad \Phi_{e,t}^{\tau_1 \oplus \tau_2}(n) \downarrow],$$

where $\tau_1 \oplus \tau_2$ is defined to be the shortest string $\rho \in 2^{<\omega}$ such that $\rho(2x) = \tau_1(x)$ and $\rho(2x+1) = \tau_2(x)$. The obvious modification of Cases 1 and 2 ensures that $A \neq \Phi_e^{B \oplus C}$, which implies that $A \oplus C \neq \Phi_e^{B \oplus C}$. At stage $2e+2$ we ensure that $B \oplus C \neq \Phi_e^{A \oplus C}$. Finally, let $\mathbf{a} = \deg(A \oplus C)$ and $\mathbf{b} = \deg(B \oplus C)$. □

generic sets

Definition 11.3

Let $\mathcal{V} = \{V_e\}_{e\in\omega}$ be a u.c.e. sequence of c.e. sets $V_e \subseteq 2^{<\omega}$, with strings identified with their indices.

1. We say $f \in 2^\omega$ *forces* V_e if it satisfies the *forcing* requirement,

$$F_e : (\exists\sigma \prec f)[\sigma \in V_e \quad \vee \quad (\forall\rho \succ \sigma)[\rho \notin V_e]]. \tag{3}$$

   If σ satisfies F_e we say that σ *forces* F_e (written $\sigma \Vdash F_e$) and any $f \succ \sigma$ also forces F_e (written $f \Vdash F_e$).

2. We say f is *generic with respect to* $\mathcal{V} = \{V_e\}_{e\in\omega}$ (written $\mathcal{V}$-generic) if f forces V_e for every $e \in \omega$.

3. We say f is *1-generic* if it is generic with respect to $\{W_e\}_{e\in\omega}$. (The term "1-generic" refers to the fact that f is deciding Σ_1 statements.)

Proposition 11.4

If A is 1-generic, then A is noncomputable.

Proof.

Given a total computable function f, we have to show that $A \neq f$. Since A is 1-generic, it suffices to define a c.e. set S such that for any set X which forces S, $X \neq f$.
For any string σ let σ^* be the string of length 1 defined by $\sigma^* = (1 - f(|\sigma|))$. Then let

$$S = \{\sigma\sigma^* : \sigma \in \{0,1\}^{<\omega}\}.$$

It is clear that S is computable (hence c.e.).
For any string $\sigma \in S$ there is an extension $\sigma\sigma^* \in S$. Thus, if X forces S, it must be the case that for some n, $X \upharpoonright (n+1) \in S$. But then $X(n) \neq f(n)$. So S has the required properties.

□

generic sets

Occasionally, we build f as the characteristic function of a set A and we wish to control the jump A'. (At a finite stage we decide whether $e \in A'$.) We can accomplish this by meeting for all e the following requirement called *forcing the jump* $\Phi_e^A(e)$:

$$J_e : \quad (\exists \sigma \prec A)[\Phi_e^\sigma(e) \downarrow \quad \vee \quad (\forall \tau \succeq \sigma)[\Phi_e^\tau(e) \uparrow]]. \tag{4}$$

Theorem 11.5 (Jockusch-Posner)

A set A is 1-generic iff A forces the jump, i.e., satisfies every jump requirement $\{J_e\}_{e \in \omega}$ of (4).

Proof.

($\Rightarrow$). Define $W_{h(e)} = \{\sigma : \Phi_e^\sigma(e) \downarrow\}$. Now A forces $W_{h(e)}$. Therefore, A forces the jump $\Phi_e^A(e)$, i.e., satisfies the requirement J_e.

($\Leftarrow$). Define a computable function $f(e)$ by

$$\Phi_{f(e)}^\sigma(z) = \begin{cases} 1 & \text{if } (\exists s \leq |\sigma|)(\exists \tau \preceq \sigma)[\tau \in W_{e,s}], \\ \uparrow & \text{otherwise.} \end{cases}$$

If A satisfies requirement $J_{f(e)}$ of (4), then we can clearly see that A forces $V_e = \{\sigma : \Phi_{f(e)}^\sigma(f(e)) \downarrow\}$ and A forces W_e.

$\square$

Theorem 11.6 (Jockusch)

There is a 1-generic set $G \leq_{\mathrm{T}} \emptyset'$.

Inverting the Jump

Theorem 11.7 (Friedberg Completeness Criterion)

For every degree $\mathbf{b} \geq \mathbf{0}'$ *there is a degree* $\mathbf{a}$ *such that* $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}' = \mathbf{b}$.

- A degree $\mathbf{a}$ is called *complete* if $\mathbf{a} \geq \mathbf{0}'$. Hence, the result also gives a criterion for $\mathbf{a}$ being complete.
- Note that for any degree $\mathbf{a}$, $\mathbf{0} \leq \mathbf{a}$ and hence $\mathbf{0}' \leq \mathbf{a}'$, i.e., any jump is above $\mathbf{0}'$. Hence, the jump, viewed as a map on degrees, has range contained in $\{\mathbf{b} : \mathbf{b} \geq \mathbf{0}'\}$.
- Theorem 11.7 asserts that this map is *onto* the set $\{\mathbf{b} : \mathbf{b} \geq \mathbf{0}'\}$.
- Theorem 11.7 also demonstrates that the jump as a map on degrees is not 1:1. To see this, apply it with $\mathbf{b} = \mathbf{0}''$ to obtain $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}' = \mathbf{0}''$. Clearly, $\mathbf{a} \mid \mathbf{0}'$, yet they have the same jump. (It is also possible to have $\mathbf{a} < \mathbf{b}$ and $\mathbf{a}' = \mathbf{b}'$.)
- However, the jump is 1:1 on sets. Because there is a computable 1:1 function f such that for any set A, $x \in A \Leftrightarrow f(x) \in A'$.

Proof of Friedberg Completeness Criterion

Fix $B \in \mathbf{b}$. We shall construct a set A, by finite initial segments $\{\sigma_s\}_{s \in \omega}$ using a B-computable finite extension construction.

Construction. *Stage* $s = 0$. Set $\sigma_0 = \emptyset$.
Stage $s + 1 = 2e + 1$. (We decide whether $e \in A'$). We meet the forcing the jump requirement J_e of (4). If A meets J_e we say that A *forces* the jump on argument e. Given σ_s, use a $\emptyset'$-oracle to test whether

$$(\exists \sigma)(\exists t)[\sigma_s \prec \sigma \quad \& \quad \Phi^{\sigma}_{e,t}(e) \downarrow]. \tag{5}$$

If (5) is satisfied, let σ be the first such σ in the standard enumeration of G_e of the Oracle Graph Theorem 3.3.8. If not, set $\sigma_{s+1} = \sigma_s$.
Stage $s + 1 = 2e + 2$. (We code $B(e)$ into A.) Let $n = |\sigma_s|$. Define $\sigma_{s+1}(n) = B(e)$.

Verification. Now $\lim_s \sigma_s$ is infinite since $|\sigma_{2e}| \geq e$. Let $A = \lim_s \sigma_s$, and $\mathbf{a} = \deg(A)$. The construction is B-computable because at odd stages we use a $\emptyset'$-oracle, at even stages we use a B-oracle, and $\emptyset' \leq_{\mathrm{T}} B$. Since $A \oplus \emptyset' \leq_{\mathrm{T}} A'$ for any A, to prove $A' \equiv_{\mathrm{T}} A \oplus \emptyset' \equiv_{\mathrm{T}} B$ it suffices to prove $A' \leq_{\mathrm{T}} B \leq_{\mathrm{T}} A \oplus \emptyset'$.

Proof of Friedberg Completeness Criterion

Lemma 11.8

$A' \leq_T B$.

Proof of Lemma 11.8.

Since the construction is B-computable, the sequence $\{\sigma_s\}_{s \in \omega}$ is B-computable. To decide whether $e \in A'$, B-computably determine using $\emptyset' \leq_T B$ whether (5) holds for $2e$. If so, $e \in A'$, and otherwise $e \notin A'$ because no $\sigma \succeq \sigma_{2e}$ makes $\Phi_e^{\sigma}(e)$ defined. □

Lemma 11.9

$B \leq_T A \oplus \emptyset'$.

Proof of Lemma 11.9.

We show $\{\sigma_s\}_{s \in \omega}$ is an $(A \oplus \emptyset')$-computable sequence. This suffices because $B(e)$ is the last value of σ_{2e+2}. The proof is by induction on s. Given σ_s ($s \leq 2e$), use a $\emptyset'$-oracle to compute σ_{2e+1}. If $n = |\sigma_{2e+1}|$ then $\sigma_{2e+2} = \sigma_{2e+1} {}^\frown A(n)$, so σ_{2e+2} is computed from σ_{2e+1} using an A-oracle. □

This completes the proof of Theorem 11.7. □

Relativized Friedberg Completeness Criterion

Theorem 11.10 (Relativized Friedberg Completeness Criterion)

For every degree $\mathbf{c}$*,*

$$F_1(\mathbf{c}): \quad (\forall \mathbf{b})[\mathbf{b} \geq \mathbf{c}' \quad \Rightarrow \quad (\exists \mathbf{a})[\mathbf{a} \geq \mathbf{c} \quad \& \quad \mathbf{a}' = \mathbf{a} \cup \mathbf{c}' = \mathbf{b}]].$$

Proof.

Do the proof of Theorem 11.7 with $\mathbf{c}$ and $\mathbf{c}'$ in place of $\mathbf{0}$ and $\mathbf{0}'$. □

Corollary 11.11

For every $n \geq 1$*, and every degree* $\mathbf{c}$*,*

$$F_n(\mathbf{c}): \quad (\forall \mathbf{b})[\mathbf{b} \geq \mathbf{c}^{(n)} \quad \Rightarrow \quad (\exists \mathbf{a})[\mathbf{a} \geq \mathbf{c} \quad \& \quad \mathbf{a}^{(n)} = \mathbf{a} \cup \mathbf{c}^{(n)} = \mathbf{b}]].$$

Proof.

To prove $(\forall \mathbf{c})F_n(\mathbf{c})$ holds for all $n \geq 1$, use induction on n and the fact that $F_{n+1}(\mathbf{c})$ follows from $F_n(\mathbf{c})$ and $F_1(\mathbf{c}^{(n)})$. □