Foundations of Computability Theory (12)

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A low simple set

Theorem 12.1

There is a simple set A which is low ($A' \equiv_T \emptyset'$).

Corollary 12.2

There is a noncomputable c.e. set A which is incomplete, i.e., $\emptyset <_T A <_T \emptyset'$.

Proof of Theorem 12.1: Requirements

We construct a coinfinite c.e. set A to meet for all e the requirements:

$$
\begin{array}{llcll}
P_e : & |W_e| = \infty & \Rightarrow & W_e \cap A \neq \emptyset; & \text{(simplicity)} \\
N_e : & (\exists^\infty s)[\Phi^{A_s}_{e,s}(e)\downarrow] & \Rightarrow & \Phi^A_e(e)\downarrow; & \text{(lowness)}
\end{array}
$$

where A_s consists of the elements enumerated in A by the end of stage s, and $A = \cup_s A_s$.

Note that the requirements $\{N_e\}_{e\in\omega}$ guarantee $A' \leq_T \emptyset'$ as follows.
Define the computable function $\hat{g}(e,s)$ by

$$
\hat{g}(e,s) := \begin{cases} 1 & \text{if} \quad \Phi^{A_s}_{e,s}(e)\downarrow; \\ 0 & \text{otherwise.} \end{cases}
$$

If requirement N_e is satisfied for all e, then $g(e) = \lim_s \hat{g}(e,s)$ exists for all e. But $g \leq_T \emptyset'$ by the Limit Lemma, and g is the characteristic function of A'. Therefore, $A' = g \leq_T \emptyset'$.

Requirements N_e are both *negative*, tending to restrain elements *out of* A, and thus there is no conflict between them, but P_e is a *positive* requirement, tending to enumerate elements *into* A. Thus we assign priority to the requirements as follows:

$$
N_0 > C_0 > P_0 > N_1 > C_1 > P_1 > N_2 > C_2 > P_2 > \dots
$$

During the construction, P_e must respect any restraint imposed by N_i for all $i \leq e$.

Proof of Theorem 12.1: Strategies

Strategy for N_e:. $\underline{(\exists^\infty s)[\Phi_{e,s}^{A_s}(e)\downarrow] \quad \Rightarrow \quad \Phi_e^A(e)\downarrow.}$

Recall that the *use function* $\varphi_{e,s}^{A_s}(x)$ is one plus the maximum element used in the computation $\Phi_{e,s}^{A_s}(x)$ if the latter converges, and is 0 otherwise. Given A_s, define for all e the *restraint function*,

$$r(e,s) := \varphi_{e,s}^{A_s}(e).$$

Now $r(e,s)$ is a computable function because $\{A_s\}_{s\in\omega}$ is a computable sequence. To meet N_e we attempt to restrain with priority N_e any elements $x \leq r(e,s)$ from entering A_{s+1}. The point is that if $\Phi_{e,s}^{A_s}(e)\downarrow$, with $r = \varphi_{e,s}^{A_s}(e)$, and N_e succeeds in preventing any $x \leq r$ from entering A later, then $A \restriction r = A_s \restriction r$ and hence $\Phi_e^A(e)\downarrow$.

Strategy for P_e:. $\underline{|W_e| = \infty \quad \Rightarrow \quad W_e \cap A \neq \emptyset.}$

When we see an element $x \in W_{e,s}$ such that $x > 2e$ and $x > r(i,s)$ for all $i \leq e$, we enumerate x into A_{s+1}, thus satisfying P_e and respect all restraints imposed by higher priority requirements N_i, $i \leq e$. By choosing $x > 2e$ we ensure that A is coinfinite.

Here x is called as a *follower* for requirement P_e.

Proof of Theorem 12.1: Construction

Construction

- *Stage* $s = 0$. Let $A_0 = \emptyset$.
- *Stage* $s + 1$. Given A_s, compute $r(e, s)$ for all $e \leq s$. Choose the least $i \leq s$ such that

$$W_{i,s} \cap A_s = \emptyset; \qquad \text{and} \tag{1}$$

$$(\exists x)[x \in W_{i,s} \quad \& \quad x > 2i \quad \& \quad (\forall e \leq i)[r(e, s) < x]]. \tag{2}$$

  If i exists, choose the least x satisfying (2). Enumerate x in A_{s+1}, and say that requirement P_i *receives attention*. If i does not exist, do nothing, and define $A_{s+1} = A_s$.
- Let $A = \cup_s A_s$. This ends the construction.

If (1) holds, we say that requirement P_i *requires attention* at stage $s + 1$. Then by (2) we are searching for a *follower* x for P_i.
If P_i receives attention at stage $s + 1$, then $W_{i,s} \cap A_{s+1} \neq \emptyset$, P_i is satisfied, and (1) fails for stages $> s + 1$. Therefore, P_i never again needs (and receives) attention.

Proof of Theorem 12.1: Verification I

We define the injure sets as $I_e = \{x : (\exists s)[x \in A_{s+1} \setminus A_s \quad \& \quad x \leq r(e,s)]\}$.

Lemma 12.3

For all e, the injury set I_e is finite. (That is, N_e is injured finitely often.)

Proof.

Each positive requirement P_i receives attention at most once, thus contributes at most one element to A. But by the third clauses of (2), N_e can be injured by P_i only if $i < e$, and at most once for each such i. Hence, $|I_e| \leq e$. □

Lemma 12.4

$(\forall e)[$*requirement* N_e *is met* $\& \ r(e) := \lim_s r(e,s)$ *exists*$]$.

Proof.

Fix e. By Lemma 12.3, choose stage s_e such that N_e is not injured at any stage $t > s_e$. If $\Phi_{e,t}^{A_t}(e)$ converges for some $t > s_e$, then by induction on $s \geq t$, $r(e,s) = r(e,t)$. Thus, $\lim_s r(e,s) = r(e,t)$ exists. Moreover, $A_s \upharpoonright r(e) = A \upharpoonright r(e)$. Therefore, $\Phi_e^A(e)$ is defined by the use principle. If $\Phi_{e,t}^{A_t}(e)$ never converges for any $t > s_e$, then $\Phi_e^A(e)$ is undefined and $\lim_s r(e,s) = 0$. □

Proof of Theorem 12.1: Verification II

Lemma 12.5

$(\forall i)[\textit{requirement } P_i \textit{ is met}]$.

Proof.

By induction, choose s such that no P_j, $j < i$, needs attention after stage s. By Lemma 12.3 and 12.4, choose $s' \geq s$ such that

$$(\forall t \geq s')(\forall e \leq i)[\, r(e,t) = r(e) \,].$$

Now suppose W_i is infinite. Choose $t > s'$ such that

$$(\exists x)[\, x \in W_{i,t} \quad \& \quad x > 2i \quad \& \quad (\forall e \leq i)[\, x > r(e) \,]\,].$$

Now either $W_{i,t} \cap A_t \neq \emptyset$ or else P_i receives attention at stage $t+1$. In either case, $W_{i,t} \cap A_{t+1} \neq \emptyset$. Therefore, P_i is met by the end of stage $t+1$. □

This completes the proof of Theorem 12.1 □

Friedberg-Muchnik Theorem

Theorem 12.6 (Friedberg, 1957; Muchnik, 1956)

There exist c.e. sets A and B of incomparable Turing degree.

We construct c.e. sets A and B to meet the same requirements as in the proof of the Kleene-Post theorem,

$$R_e : A \neq \Phi_e^B \quad \& \quad Q_e : B \neq \Phi_e^A.$$

Note that here R_e is both positive and negative. It tends to enumerate elements *into* A, while restrain elements out of B. The same situation applies to Q_e.
We assign priority to the requirements as follows:

$$R_0 > Q_0 > R_1 > Q_1 > R_2 > Q_2 > \dots$$

During the construction, for each requirement R_e and Q_e, we define their restraint functions as $r(e, s)$ and $q(e, s)$ respectively, to prevent lower priority requirements from injuring them.

Proof of Friedberg-Muchnik Theorem: Strategy

Basic Module Strategy for R_e:. $\underline{A \neq \Phi_e^B.}$
We pick x not yet in A as follower for R_e. When R_e *requires attention* at stage $s+1$, i.e.,

$$\Phi_{e,s}^{B_s}(x) \downarrow = 0 = A_s(x).$$

We enumerate x into A, say R_e *receives attention*, and define a restraint function $r(e, s+1) = s$ which prevents lower priority requirements $\{Q_j\}_{j \geq e}$ from enumerating any $z \leq r(e, s+1)$ into B.

Let $\omega^{[y]} = \{\langle x, y\rangle : x \in \omega\}$. To avoid conflict between requirements, we choose follower $x \in \omega^{[e]}$ to meet R_e. Also note that the follower x for R_e should respect the restraint functions $q(i, s)$ for all $i < e$ imposed by higher priority requirements Q_i.

- If the higher priority requirements $\{Q_i\}_{i<e}$ have finished acting by stage s then requirement R_e is met because the restraint function $r(e, s+1)$ ensures that $B \upharpoonright s = B_s \upharpoonright s$, so that

$$\Phi_e^B(x) \downarrow = \Phi_{e,s}^{B_s}(x) = 0 \quad \neq \quad 1 = A(x)$$

- If some Q_i, $i < e$, acts at some stage $t > s+1$, R_e is injured. Then we let $r(e, t) = 0$, reset the R_e basic module with a fresh witness x' and begin all over.

$r(e, s) > 0$ indicates that R_e has been satisfied at some stage $t \leq s$ and not injured since then. On the other hand, $r(e, s) = 0$ indicates that R_e has either never been satisfied, or satisfied, injured, and never satisfied since then until now.

Proof of Friedberg-Muchnik Theorem: Construction

During the construction, for all i and s, if not mentioned, the values of $r(i,s)$ and $q(i,s)$ stay unchanged and pass to the next stage.
Stage $s = 0$. Define $r(e,s) = q(e,s) = 0$ for all e.

Stage $s+1 = 2k+1$. Choose the least e such that

$$r(e,s) = 0 \quad \& \quad (\exists x < s)[\, x \in \omega^{[e]} \setminus A_s \quad \& \quad (\forall i < e)[\, x > q(i,s)\,] \quad \& \quad \Phi_{e,s}^{B_s}(x)\downarrow = 0\,].$$

If there is no such e, then do nothing and go to stage $s+2$. Otherwise, choose the least such e and the least corresponding x. We perform the following steps, and say R_e *acts* at stage $s+1$.

1. Enumerate x in A. (This makes $A(x) = A_{s+1}(x) = 1$.)
2. Define $r(e, s+1) = s$. (This attempts to restrain $B_s \upharpoonright s$ in order to preserve the computation $\Phi_{e,s}^{B_s}(x)\downarrow = 0 \neq 1 = A_{s+1}(x)$.)
3. For all $i \geq e$, define $q(i, s+1) = 0$. We say that these lower priority requirements $\{Q_i\}_{i \geq e}$ are *injured* at stage $s+1$ and are *reset*.

Stage $s+1 = 2k+2$. Do the same for e with the roles of A, R, r and B, Q, q reversed and with respect to the priority ordering.

Proof of Friedberg-Muchnik Theorem: Verification I

Lemma 12.7

If requirement R_e acts at some stage $s+1$ and is never later injured, then requirement R_e is met and $r(e,t) = s$ for all $t \geq s+1$. The same holds for Q_e.

Proof.

Suppose R_e acts at stage $s+1$. Then $\Phi_{e,s}^{B_s}(x) \downarrow = 0$ for some $x \in A_{s+1}$. Since it is never later injured, by induction $r(e,t) = r(e,s) = s$ for all $t \geq s+1$. Moreover, as no number less than s is enumerated into B after stage $s+1$, $B \restriction s = B_s \restriction s$, and

$$\Phi_e^B(x) \downarrow = \Phi_{e,s}^{B_s}(x) \downarrow = 0 \neq 1 = A(x).$$

Therefore, R_e is met. □

Proof of Friedberg-Muchnik Theorem: Verification II

Lemma 12.8

For every e, requirement R_e and Q_e act at most finitely often, and $r(e) = \lim_s r(e,s)$, $q(e) = \lim_s q(e,s)$ exists.

Proof.

We prove by induction on the priority ordering. Without loss of generality, assume true for all R_i and Q_i, $i < e$. Now we show it is true for R_e.
Let v be the greatest stage at which some Q_i acts for $i < e$, if ever, and $v = 0$ if none exists. Then $r(e,v) = 0$ and this will persist until some stage $s+1 > v$, if ever, when R_e acts. If R_e acts at some stage $s+1 > v$ then by Lemma 12.7 that action satisfies R_e, which never acts again, and $r(e,t) = s$ for all $t \geq s+1$. Either way, $r(e)$ exists and R_e acts at most finitely often. □

Proof of Friedberg-Muchnik Theorem: Verification III

Lemma 12.9

For every e, requirement R_e and Q_e are met.

Proof.

Now suppose that R_e is not met. Then $\Phi_e^B = A$. By Lemma 12.8, let v be the least stage such that for $i \leq e$, no Q_i or R_i acts after stage v, and the limits $r(i,s) = r(i)$ and $q(i,s) = q(i)$ for all $s \geq v$.
Now by stage v at most finitely many elements $x \in \omega^{[e]}$ have been enumerated in A. Choose the least $x \in \omega^{[e]} - A_v$ such that $x > v$. Eventually, there will be a stage $s > x$ such that $\Phi_{e,s}^{B_s}(x) \downarrow = 0$ since $x \notin A_s$. Hence, R_e requires attention at stage $s+1$ and will act at this stage, which is a contradiction. Thus, R_e is met. □

This completes the proof of the Friedberg-Muchnik Theorem □