Foundations of Computability Theory (13)

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Cone Avoidance Theorem

Theorem 13.1 (Cone Avoidance Theorem, Sacks)

For every noncomputable c.e. set C there is a simple set A such that $C \not\leq_T A$ (and hence $\emptyset <_T A <_T \emptyset'$).

It suffices to construct A to be coinfinite and to satisfy, for all e, the requirements:

$$P_e: \quad |W_e| = \infty \quad \Rightarrow \quad W_e \cap A \neq \emptyset; \qquad \text{(simplicity)} \qquad (1)$$

$$N_e: \quad \Phi_e^A \neq C. \qquad \text{(cone avoidance)} \qquad (2)$$

Let $\{C_s\}_{s\in\omega}$ be a computable enumeration of C. We shall give a computable enumeration $\{A_s\}_{s\in\omega}$ of A. Given $\{C_t\}_{t\leq s}$ and $\{A_t\}_{t\leq s}$, define for all e the following length function $\ell(e,s)$ and restraint function $r(e,s)$.

$$\ell(e,s) := \max\left\{x : (\forall y < x)\left[\Phi_e^A(y)[s] \downarrow = C_s(y)\right]\right\}; \qquad (3)$$

$$r(e,s) := \max\left\{\varphi_e^A(x)[s] : x \leq \ell(e,s)\right\}. \qquad (4)$$

We assign priority to the requirements as follows:

$$N_0 > P_0 > N_1 > P_1 > N_2 > P_2 > \cdots$$

Cone Avoidance Theorem: Construction

Stage $s = 0$. Define $A_0 = \emptyset$.

Stage $s + 1$. Given A_s, compute $\ell(e, s)$ and $r(e, s)$ as above for all $e \leq s$. We say P_i *requires attention* at stage $s + 1$ if $i \leq s$, $W_{i,s} \cap A_s = \emptyset$, and

$$(\exists x)[x \in W_{i,s} \quad \& \quad x > 2i \quad \& \quad (\forall e \leq i)[r(e, s) < x]]. \tag{5}$$

If i exists, choose the least such i and the least corresponding x, and enumerate x in A_{s+1}. We say P_i *receives attention (acts)* at stage $s + 1$.

Remark I.

- Requirement N_e restrains with priority N_e any $x \leq r(e, s)$ from entering $A_{s+1} - A_s$.
- If $x \leq r(e, s)$ and $x \in A_{s+1} - A_s$ then x represents an *injury* to requirement N_e at stage $s + 1$.
- Requirement N_e can be injured at most e times, because it is injured at most once by each P_i, $i < e$, whereupon P_i is satisfied forever.
- Define the injury set $I_e = \{x : (\exists s)[x \in A_{s+1} - A_s \quad \& \quad x \leq r(e, s)]\}$. Then $|I_e| \leq e$.

Proof of Cone Avoidance Theorem: Verification I

Remark II.

Fix e. Assume that $z \in A_{s+1} - A_s$ and no $P_j, j < e$, receives attention at stage $s+1$.

1. For $y < \ell(e,s)$, by the definition of $\ell(e,s)$,

$$\Phi_e^A(y)[s] = C_s(y).$$

   On the other hand, by construction $z > \varphi_e^A(y)[s]$. So

$$\Phi_e^A(y)[s+1] \downarrow = \Phi_e^A(y)[s] = C_s(y) \quad \text{and} \quad \varphi_e^A(y)[s+1] = \varphi_e^A(y)[s].$$

2. For $y = \ell(e,s)$, by the definition of $\ell(e,s)$,

$$\Phi_e^A(y)[s] \neq C_s(y).$$

   On the other hand, by construction $z > \varphi_e^A(y)[s]$. So if $\varphi_e^A(y)[s] \downarrow$ then

$$\Phi_e^A(y)[s+1] \downarrow = \Phi_e^A(y)[s] \neq C_s(y) \quad \text{and} \quad \varphi_e^A(y)[s+1] = \varphi_e^A(y)[s].$$

Proof of Cone Avoidance Theorem: Verification I

Lemma 13.2

$(\forall e)[\Phi_e^A \neq C]$.

Proof.

Assume for a contradiction that $\Phi_e^A = C$. Then $\lim_s \ell(e,s) = \infty$. Choose s' such that N_e is never injured after stage s'. We shall define a procedure to compute C, contrary to hypothesis. To compute $C(p)$ for some $p \in \omega$ find the least $s > s'$ such that $\ell(e,s) > p$. It follows by induction on $t \geq s$ that

$$(\forall t \geq s)[\ell(e,t) > p \quad \& \quad r(e,t) \geq \max\{\varphi_e^A(x)[s] : x \leq p\}]. \tag{6}$$

and hence that $\Phi_e^A(p)[s] = \Phi_e^{A_s}(p) = \Phi_e^A(p) = C(p)$, whence C is computable.

To prove (6), assume it holds for $t \geq s$. Then for any $x \leq p$, by Remark II, $t > s'$ ensure that $A_{t+1} \restriction z = A_t \restriction z$ for all numbers z used in a computation $\Phi_e^A(x)[t] = y$. Thus, $\max\{\varphi_e^A(x)[t+1] : x \leq p\} = \max\{\varphi_e^A(x)[t] : x \leq p\}$ and $\Phi_e^A[t+1] \restriction p = \Phi_e^A[t] \restriction p$. The latter also implies $\ell(e,t+1) > p$, unless $C_{t+1}(x) \neq C_t(x)$ for some $x < \ell(e,t)$. But if $C_{t+1}(x) \neq C_t(x)$ for some $t \geq s$ and $x \leq p$, where x is minimal, then by Remark II, the disagreement $C_t(x) \neq \Phi_e^{A_t}(x)$ is preserved forever, contrary to our hypothesis that $C = \Phi_e^A$. □

Proof of Cone Avoidance Theorem: Verification II

Lemma 13.3

$(\forall e)\ [\lim_s r(e,s) < \infty]$.

Proof.

By Lemma 13.2, choose $p = (\mu x)[C(x) \neq \Phi_e^A(x)]$. Choose s' sufficiently large such that for all $s \geq s'$,

- $(\forall x < p)[\Phi_e^{A_s}(x)\downarrow = \Phi_e^A(x)]$;
- $(\forall x \leq p)[C_s(x) = C(x)]$; and
- N_e is not injured at stage s.

Case 1. $(\forall s \geq s')[\Phi_e^{A_s}(p)\uparrow]$. Then $r(e,s) = r(e,s')$ for all $s \geq s'$.
Case 2. $\Phi_e^{A_t}(p)\downarrow$ for some $t \geq s'$. Then $\Phi_e^A(p)[s] = \Phi_e^A(p)[t]$ for all $s \geq t$ because $\ell(e,s) \geq p$, and so, by the definition of $r(e,s)$, and the fact that N_e is not injured after stage t, the computation $\Phi_e^{A_t}(p)$ is preserved forever. Thus, $\Phi_e^A(p) = \Phi_e^A(p)[s]$. But $C(p) \neq \Phi_e^A(p)$. Hence,

$$(\forall s \geq t)[C_s(p) \neq \Phi_e^A(p)[s]\ \&\ \ell(e,s) = p\ \&\ r(e,s) = r(e,t)].$$

Therefore, $\lim_s r(e,s) = r(e,t)$. □

Proof of Cone Avoidance Theorem: Verification III

Lemma 13.4

$(\forall e)[W_e \text{ } \textit{infinite} \quad \Rightarrow \quad W_e \cap A \neq \emptyset]$. $(\forall e)[\textit{requirement } P_e \textit{ is met}]$.

Proof.

We prove by induction on e. Suppose all requirements P_i, $i < e$, are met. By Lemma 13.3, choose s' sufficiently large such that no P_i, $i < e$, requires attention after stage s', and such that $r(i,s) = r(i,s')$ for all $s \geq s'$ and $i \leq e$.
Now suppose W_e is infinite. Choose $t > s'$ such that

$$(\exists x)[\, x \in W_{e,t} \quad \& \quad x > 2e \quad \& \quad (\forall i \leq e)[\, x > r(i,t)\,]\,].$$

Now either $W_{e,t} \cap A_t \neq \emptyset$ or else P_e receives attention at stage $t+1$. In either case, $W_{e,t} \cap A_{t+1} \neq \emptyset$. Therefore, P_e is met by the end of stage $t+1$. □

By Remarks I, every N_e is met. By Lemma 13.4, every P_e is met. Note that $\bar{A}$ is infinite by the clause "$x > 2e$" in (5) and hence A is simple. This completes the proof of the Theorem. □

Sacks Splitting Theorem

Theorem 13.5 (Sacks Splitting Theorem)

Let B and C be c.e. sets such that C is noncomputable. Then there exist low c.e. sets A_0 and A_1 such that:

1. $A_0 \cup A_1 = B$ *and* $A_0 \cap A_1 = \emptyset$, *and*
2. $C \not\leq_{\mathrm{T}} A_i$, *for* $i = 0, 1$.

Sacks Splitting Theorem: Corollaries

Proposition 13.6

If the c.e. set B is the disjoint union of c.e. sets A_0 and A_1 then $B \equiv_T A_0 \oplus A_1$.

Proof.

First, note that $A_i \leq_{\mathrm{T}} B$ for $i = 0, 1$, because to decide whether $x \in A_i$, first ask whether $x \in B$. If so, enumerate A_0 and A_1 until x appears in one of them. Hence, $A_0 \oplus A_1 \leq_{\mathrm{T}} B$. In the other direction, obviously $B \leq_{\mathrm{T}} A_0 \oplus A_1$. □

Corollary 13.7

If $\mathbf{b}$ *is any nonzero c.e. degree, there are incomparable low c.e. degrees* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *such that* $\mathbf{b} = \mathbf{a}_0 \cup \mathbf{a}_1$.

Proof.

Choose a c.e. set $B \in \mathbf{b}$. Apply the Sacks Splitting Theorem 13.5 to B with $C = B$ to obtain A_0 and A_1. Hence, $A_0, A_1 <_{\mathrm{T}} B$. By Proposition 13.6, $B \equiv_{\mathrm{T}} A_0 \oplus A_1$. Now A_0 and A_1 cannot have comparable degree because if, say, $A_0 \leq_{\mathrm{T}} A_1$ then $A_1 \equiv_{\mathrm{T}} B$. Let $\mathbf{a}_0 = \deg(A_0)$ and $\mathbf{a}_1 = \deg(A_1)$, then $\mathbf{b} = \mathbf{a}_0 \cup \mathbf{a}_1$. □

Sacks Splitting Theorem: Corollaries

Corollary 13.8

The low c.e. degrees generate the c.e. degrees when closed under join.

Corollary 13.9

No c.e. degree is minimal.

Corollary 13.10

For any c.e. degree $\mathbf{c}$, $\mathbf{0} < \mathbf{c} < \mathbf{0}'$, *there is a c.e. degree incomparable with* $\mathbf{c}$.

Proof.

Let $B = K$, let $C \in \mathbf{c}$ be c.e., and apply the Sacks Splitting Theorem 13.5. One of A_0 and A_1 must have degree incomparable with $\mathbf{c}$, or else $A_0 \leq_{\mathrm{T}} C$ and $A_1 \leq_{\mathrm{T}} C$. Hence, $K \leq_{\mathrm{T}} C$, contrary to $C <_{\mathrm{T}} \emptyset'$. $\square$

Proof of Sacks Splitting Theorem: Requirements

We may assume that B is infinite, or else the result is trivial.
Let $\{B_s\}_{s\in\omega}$ and $\{C_s\}_{s\in\omega}$ be computable enumerations of B and C such that $B_0 = \emptyset$ and $|B_{s+1} - B_s| \leq 1$ for all s. We shall give computable enumerations $\{A_{i,s}\}_{s\in\omega}$, $i = 0, 1$, satisfying the single positive requirement

$$P: \quad x \in B_{s+1} - B_s \quad \Rightarrow \quad [x \in A_{0,s+1} \quad \text{or} \quad x \in A_{1,s+1}],$$

and the negative requirements for $i = 0, 1$, and all e,

$$N_{\langle e,i\rangle}: \quad C \neq \Phi_e^{A_i}.$$

(The lowness of A_i will follow from the strategy to achieve the $N_{\langle e,i\rangle}$ requirements, because the restraint function $r^i(e, s)$ exists and is finite.)

Proof of Sacks Splitting Theorem: Construction

For $i = 0, 1$, and all e, s, define the computable functions $\ell^i(e,s)$ and $r^i(e,s)$ as before.

$$\ell^i(e,s) := \max\left\{x : (\forall y < x)\left[\Phi_e^{A_i}(y)[s] \downarrow = C_s(y)\right]\right\};$$
$$r^i(e,s) := \max\left\{\varphi_e^{A_i}(x)[s] : x \le \ell^i(e,s)\right\}.$$

Construction

Stage $s = 0$. Define $A_{i,0} = \emptyset$, $i = 0, 1$.
Stage $s + 1$. Given $A_{i,s}$ and $x \in B_{s+1} - B_s$, choose $\langle e', i' \rangle$ to be the least $\langle e, i \rangle \le s$ such that $x \le r^i(e,s)$, and enumerate x in $A_{1-i',s+1}$. (Namely, choose the highest priority requirement $N_{e,i'}$ which would be injured by enumerating x in $A_{i'}$, and enumerate x on the "other side" $A_{1-i'}$.) If $\langle e', i' \rangle$ fails to exist, enumerate x into $A_{0,s+1}$.

Proof of Sacks Splitting Theorem: Verification I

To see that the construction succeeds, define the injury set I_e^i as before:

$$I_e^i = \{x : (\exists s)[x \in A_{i,s+1} - A_{i,s} \quad \& \quad x \leq r^i(e,s)]\}$$

It follows by induction on $\langle e, i \rangle$ that for $i = 0, 1$, and all e,

(i) I_e^i is finite;

(ii) $C \neq \Phi_e^{A_i}$;

(iii) $r^i(e) := \lim_s r^i(e,s)$ exists and is finite.

Namely, fix $\langle e, i \rangle$ and assume (i), (ii), and (iii) hold for all $\langle k, j \rangle < \langle e, i \rangle$. By (iii), choose t such that $r^j(k,s) = r^j(k)$ for all $\langle k, j \rangle < \langle e, i \rangle$ and all $s \geq t$. Choose r greater than all such $r^j(k)$. Choose $v > t$ with $B \upharpoonright r = B_v \upharpoonright r$. Now $N_{e,i}$ is never injured after stage v, so (i) holds for $\langle e, i \rangle$. Now (ii) and (iii) hold for $\langle e, i \rangle$ exactly as in Lemmas 13.2 and 13.3.

Proof of Sacks Splitting Theorem: Verification II

Lemma 13.11

A_0 and A_1 are both low.

Proof.

To see that each A_i is low, define the computable function g as follows:

$$\Phi^X_{g(e)}(y) = \begin{cases} C(0) & \text{if } y = 0 \ \& \ \Phi^X_e(e)\downarrow \\ \uparrow & \text{otherwise.} \end{cases}$$

Note that for $i = 0, 1$,

$$e \in A_i' \quad \Leftrightarrow \quad \Phi^{A_i}_e(e)\downarrow \quad \Leftrightarrow \quad \Phi^{A_i}_{g(e)}(0) = C(0) \quad \Leftrightarrow \quad \lim_s \ell^i(g(e), s) > 0,$$

so $A_i' \in \Delta^0_2$ because ℓ^i and g are computable functions, and $\lim_s \ell^i(g(e), s)$ exists and hence $A_i' \leq_{\mathrm{T}} \emptyset'$. □

This completes the proof of the Sacks Splitting Theorem. □

Note that the injury set I^i_e, although finite, has no obvious bounding function. Indeed, in general there is no computable function g such that $|I^i_e| \leq g(i, e)$.