Foundations of Computability Theory (15)

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Σ_1-completeness of PA

Theorem 15.1 (Gödel)

Given a computably enumerable theory T in the language $\mathcal{L}_{PA}$, the set of formulas F such that $T \vdash F$ is computably enumerable.

It suffices to do a search on all the possible proofs from the axioms of T and to list all the formulas they prove.

Lemma 15.2

If $\mathbb{N} \models F$ where F is a closed Σ_1 formula, then $PA \vdash F$.

Proof.

Let F be a formula of the form $\exists x G(x)$ for a Δ_0 formula G. If F is true in $\mathbb{N}$ then there is an integer $n \in \mathbb{N}$ such that $G(n)$ is true in $\mathbb{N}$. So it boils down to show that if a Δ_0 formula is true in $\mathbb{N}$, then it is provable in PA. This can be shown by induction on the complexity of the formula, i.e. , how they're built up using the logical connectives and the bounded quantifiers from atomic formulas. For the bounded quantifiers, for a bounded existential quantification, it suffices to take a witness integer for this quantification and to show the formula with this witness, and for a universal quantification bounded by an integer a, it suffices to show the formula for each integer less than a. □

Gödel's first incompleteness theorem

Theorem 15.3 (Gödel's first incompleteness theorem)

Let $T \supseteq \mathrm{PA}$ be a consistent c.e. theory, such that if T proves a Σ_1 formula, then $\mathbb{N}$ is a model of this formula. Then, there exists a Σ_1 formula F such that $T \nvdash F$ and $T \nvdash \neg F$.

Proof.

According to Theorem 14.7 there exists a Σ_1 formula $F(e)$ of $\mathcal{L}_{\mathrm{PA}}$ such that

$$\{e \in \mathbb{N} : \mathbb{N} \models F(e)\} = \{e \in \mathbb{N} : \exists t\, \Phi_e(e)[t] \downarrow\}.$$

Suppose that for all e, we have $T \vdash F(e)$ or $T \vdash \neg F(e)$. Note that if $\exists t\, \Phi_e(e)[t] \downarrow$ then $\mathbb{N} \models F(e)$ and therefore $\mathrm{PA} \vdash F(e)$ according to Lemma 15.2. As $\mathrm{PA} \subseteq T$ we also have $T \vdash F(e)$.

Now, if $\forall t\, \Phi_e(e)[t] \uparrow$ we have $\mathbb{N} \models \neg F(e)$. Then we cannot have $T \vdash F(e)$ because we would then have $\mathbb{N} \models F(e)$. By our assumption that T is complete, we therefore have $T \vdash \neg F(e)$.

It follows that the c.e. set $\{e \in \mathbb{N} : T \vdash \neg F(e)\}$ coincides with the set $\{e \in \mathbb{N} : \forall t\, \Phi_e(e)[t] \uparrow\}$ which makes the complement of the halting problem a c.e. set. Contradiction. $\square$

DNC_2 functions

Recall that a function g is DNC if $g(n) \neq \Phi_n(n)$ for all n such that $\Phi_n(n) \downarrow$.

Definition 15.4

Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $2 \leq f(n) \leq f(n+1)$. A set X is of *DNC_f* degree if X computes a function g such that $g(n) < f(n)$ and $g(n) \neq \Phi_n(n)$ for all n such that $\Phi_n(n) \downarrow$.

DNC_2 stands for DNC_f where the function $f = 2$. Note that it is the strongest possible of this order: the computed function f has only two possibilities (0 or 1) to differ from each $\Phi_n(n)$.

Gödel-Rosser's incompleteness theorem

Theorem 15.5 (Gödel-Rosser's incompleteness theorem)

Let $T \supseteq \mathrm{PA}$ be a consistent c.e. theory. There exists a Σ_1 formula F such that $T \nvdash F$ and $T \nvdash \neg F$.

Proof.

According to Theorem 14.7, there exist Σ_1 formulas $F_0(e)$ and $F_1(e)$ such that

$$\mathbb{N} \models F_0(e) \quad \Leftrightarrow \quad \exists t\, \Phi_e(e)[t] \downarrow = 0, \quad \text{and} \quad \mathbb{N} \models F_1(e) \quad \Leftrightarrow \quad \exists t\, \Phi_e(e)[t] \downarrow = 1.$$

Assume $T \vdash F$ or $T \vdash \neg F$ for any Σ_1 formula F. Then for each e, either $T \vdash F_0(e)$ or $T \vdash \neg F_0(e)$. We can then enumerate using Theorem 15.1 all the formulas demonstrated by T, until we find $F_0(e)$ or $\neg F_0(e)$. Let

$$f(e) = \begin{cases} 1 & \text{if } T \vdash F_0(e); \\ 0 & \text{if } T \vdash \neg F_0(e). \end{cases}$$

Note that PA also proves $F_0(e) \to \neg F_1(e)$ and $F_1(e) \to \neg F_0(e)$: in other words $e \mapsto \Phi_e(e)$ is a partial function which cannot have both 0 and 1 as a value for the same element. By the following observation, f is a DNC_2 function, which contradicts to the fact that f is computable.

$$\Phi_e(e) \downarrow = 1 \quad \Rightarrow \quad \mathbb{N} \models F_1(e) \quad \Rightarrow \quad T \vdash F_1(e) \quad \Rightarrow \quad T \nvdash F_0(e) \quad \Rightarrow \quad f(e) = 0 \neq \Phi_e(e);$$

$$\Phi_e(e) \downarrow = 0 \quad \Rightarrow \quad \mathbb{N} \models F_0(e) \quad \Rightarrow \quad T \vdash F_0(e) \quad \Rightarrow \quad f(e) = 1 \neq \Phi_e(e).$$

Gödel-Rosser's incompleteness theorem

Corollary 15.6

Let $T \supseteq \mathrm{PA}$ be a complete and consistent theory. Then T computes a DNC_2 set.

Proof.

From the proof of the previous theorem. $\square$

Gödel's second incompleteness theorem

For a computably enumerable theory T in $\mathcal{L}_{PA}$, let $\mathrm{Coh}(T)$ the closed Π_1 formula corresponding to "T is a consistent theory", that is to say "for any proof p in T, p is not a proof of $\perp$".
The existence of such a formula follows from the correspondence between Σ_n^0/Π_n^0 sets and Σ_n/Π_n formulas.

Theorem 15.7 (3.12 Gödel's second incompleteness theorem)

Let T be a c.e. consistent theory containing PA. Then, $T \nvdash Coh(T)$.

Notation: $\ulcorner S \urcorner$ indicate the transformation of a statement S into a statement of arithmetic.

Proof of Gödel's second incompleteness theorem

The first step is to see that the proof of Theorem 15.5 can be formalized, via an appropriate coding, in Peano arithmetic: there exists a Σ_1 formula F such that

$$\mathrm{PA} \vdash \mathrm{Coh}(T) \rightarrow (\ulcorner T \nvdash F \urcorner \wedge \ulcorner T \nvdash \neg F \urcorner).$$

Let us assume by the absurdity that $T \vdash \mathrm{Coh}(T)$. Then, by the Modus Ponens rule, we have $T \vdash \ulcorner T \nvdash F \urcorner \wedge \ulcorner T \nvdash \neg F \urcorner$ and therefore $T \vdash \ulcorner T \nvdash \neg F \urcorner$ as well as $T \vdash \ulcorner T \nvdash F \urcorner$.
It should then be noted that the proof of Lemma 15.2 can also be done in Peano arithmetic, that is to say that Peano arithmetic shows that if a closed Σ_1 formula is true, then it is demonstrable in Peano arithmetic - and therefore in T. This therefore gives formally with the formula F

$$T \vdash F \rightarrow \ulcorner T \vdash F \urcorner.$$

We then have by contraposition (using the equivalence between the negation and the encoding of the negation):

$$T \vdash \ulcorner T \nvdash F \urcorner \rightarrow \neg F.$$

As $T \vdash \ulcorner T \nvdash F \urcorner$ then by Modus Ponens we get

$$T \vdash \neg F.$$

Finally, it suffices to see that if T proves any formula, then PA (and therefore T) shows that T proves this formula. This is once again an application of the formalization of the proof of Lemma 15.2 in T, the sentence $T \vdash F$ being Σ_1. So we finally have

$$T \vdash \ulcorner T \vdash \neg F \urcorner \quad \text{which contradicts } T \vdash \ulcorner T \nvdash \neg F \urcorner.$$

PA degrees

Let us fix a computable enumeration $\psi_0, \psi_1, \psi_2, \ldots$ of all the formulas of arithmetic. Suppose also that there exists a computable function $\mathrm{neg} : \mathbb{N} \to \mathbb{N}$ such that $\psi_{\mathrm{neg}(n)} = \neg\psi_n$.

Definition 15.8

A *completion of Peano arithmetic* is a complete theory T containing $\{n \in \mathbb{N} : \mathrm{PA} \vdash \psi_n\}$. A Turing degree is *PA* if it contains a consistent completion of Peano arithmetic.

The PA degrees being upward-closed, it is equivalent for a degree to be PA and to contain a set which computes a consistent completion of Peano arithmetic.

PA degrees

Theorem 15.9 (Jockusch & Soare; Solovay)

A set computes completion of PA iff it computes a DNC_2 function.

"$\Rightarrow$:" by Corollary 15.6.
"$\Leftarrow$:" let $f \leq_T X$ be a $\{0,1\}$-valued function such that $f(n) \neq \Phi_n(n)$ for all n. We are going to define a uniformly X-computable increasing sequence of consistent theories such that $\mathrm{PA} = T_0 \subseteq T_1 \subseteq \ldots$ and $T = \bigcup_n T_n$ is complete.
Let $\mathcal{T}(W_y)$ be the theory whose axioms are the formulas of code in W_y. By the s-m-n theorem we define two computable functions $\hat{s}$ and h as follows.

$$\Phi_{\hat{s}(x,y)}(z) = \Psi(x,y,z) = \begin{cases} 1 & \text{if } \mathcal{T}(W_x) + \psi_y \vdash 0 = 1; \\ 0 & \text{if } \mathcal{T}(W_x) + \neg\psi_y \vdash 0 = 1; \\ \uparrow & \text{otherwise.} \end{cases}$$

$$\Phi_{h(x,y)}(z) = \Psi'(x,y,z) = \begin{cases} 1 & \text{if } \Phi_x(z)\downarrow \text{ or } z = y; \\ \uparrow & \text{otherwise.} \end{cases}$$

Note that we have $W_{h(x,y)} = W_x \cup \{y\}$.

PA degrees

As #PA is computable, there exists e_0 such that $\mathcal{T}(W_{e_0}) = \mathrm{PA}$. Let $T_0 = \mathrm{PA} = \mathcal{T}(W_{e_0})$. Now suppose $T_n = \mathcal{T}(W_{e_n})$ is a consistent c.e. theory. We consider the arithmetic formula ψ_n of code n. Let $s(n) = \hat{s}(e_n, n)$.

Then

$$\Phi_{s(n)}(s(n)) = \Psi(e_n, n, s(n)) = \begin{cases} 1 & \text{if } T_n + \psi_n \vdash 0 = 1; \\ 0 & \text{if } T_n + \neg\psi_n \vdash 0 = 1; \\ \uparrow & \text{otherwise.} \end{cases}$$

That is to say

$$\begin{array}{lclcl} \Phi_{s(n)}(s(n)) \downarrow= 0 & \Rightarrow & T_n + \neg\psi_n \text{ is inconsistent} & \Rightarrow & T_n + \psi_n \text{ is consistent;} \\ \Phi_{s(n)}(s(n)) \downarrow= 1 & \Rightarrow & T_n + \psi_n \text{ is inconsistent} & \Rightarrow & T_n + \neg\psi_n \text{ is consistent;} \\ \Phi_{s(n)}(s(n)) \uparrow & \Rightarrow & T_n + \psi_n \text{ and } T_n + \neg\psi_n \text{ are both consistent.} & & \end{array}$$

We define

$$e_{n+1} = \begin{cases} h(e_n, n) & \text{if } f(s(n)) = 1; \\ h(e_n, \mathrm{neg}(n)) & \text{if } f(s(n)) = 0. \end{cases} \quad \text{and} \quad T_{n+1} = \mathcal{T}(W_{e_{n+1}}).$$

As $f(s(n)) \neq \Phi_{s(n)}(s(n))$, in either case T_{n+1} is a consistent c.e. theory. The theory $T = \bigcup_n T_n$ is therefore consistent and by construction it is also complete. $\square$

PA degrees

Note that $\emptyset'$ can compute a DNC_2 function and is therefore of PA degree. The following proposition implies that it is not at all necessary to be Turing complete to compute a complete and consistent extension of Peano arithmetic.

Definition 15.10

The *degree spectrum* of a class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ is the set

$$\deg \mathcal{P} = \{\deg_T X : X \in [P]\}$$

Proposition 15.11

There is a Π^0_1 class whose degree spectrum corresponds to the PA degrees.

Proof.

This is a simple observation, which was already used for the proof of Proposition 8.13. The class of DNC_2 sets is described as follows:

$$\mathcal{P} = \{X \in 2^\omega : \forall e \, \forall t \; (\Phi_e(e)[t] \uparrow \vee \Phi_e(e)[t] \downarrow \neq X(e))\} .$$

□

PA degrees

Corollary 15.12

There are low PA degrees.

Corollary 15.13

There are computably dominated PA degrees.

Corollary 15.14

Let A be a non-computable set. There exists a PA degree which does not compute A.