Foundations of Computability Theory (2)

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c.e. sets

Definition 2.1

- A set $A \in \mathbb{N}$ in *computable* if its characterization function χ_A is computable.

$$\chi_A(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

- A set $A \in \mathbb{N}$ in *computably enumerable (c.e.)* if it is the domain of some partial computable function.

We also use $A(x)$ to denote $\chi_A(x)$.

properties of c.e. sets

Proposition 2.2

A is computable iff A and $\overline{A}$ are c.e.

Proof.

"$\Rightarrow$": If A is computable, then the following two functions are partial computable, and $\mathrm{dom}(f) = A, \mathrm{dom}(g) = \overline{A}$.

$$f(x) = \begin{cases} 1 & \text{if } A(x) = 1; \\ \uparrow & \text{otherwise.} \end{cases} \qquad g(x) = \begin{cases} 1 & \text{if } A(x) = 0; \\ \uparrow & \text{otherwise.} \end{cases}$$

"$\Leftarrow$": Suppose there are partial computable functions f and g such that $\mathrm{dom}(f) = A, \mathrm{dom}(g) = \overline{A}$. Define

$$h(x) = \begin{cases} 1 & \text{if } f(x)\downarrow; \\ 0 & \text{if } g(x)\downarrow. \end{cases}$$

Then h is a total computable function and $h = \chi_A$. □

Proposition 2.3

If A, B are c.e. sets, then $A \cup B$ and $A \cap B$ are also c.e. sets.

c.e. approximations

Definition 2.4

Let the eth c.e. set be denoted by

$$W_e := \mathrm{dom}(\Phi_e) = \{x \colon \Phi_e(x) \downarrow\}.$$

A sequence of sets $A_0, A_1, \ldots$ is *uniformly computable* if there exists a computable function $f \colon \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ such that $f(n, x) = A_n(x)$.

Definition 2.5

A *c.e. approximation* of a set A is a uniformly computable sequence of sets $A_0, A_1, \ldots$ such that $A_n \subseteq A_{n+1}$ for all n, and $\bigcup_n A_n = A$.

Notation 2.6

Let $\Phi_{e,s}(x) \downarrow = y$ denote that $x, y, e < s$ and on input x in $< s$ steps Turing machine P_e halts and outputs y. Otherwise we write $\Phi_{e,s}(x) \uparrow$. Sometimes, we also write $\Phi_e(x)[s]$ instead of $\Phi_{e,s}(x)$. Let $W_{e,s} := \mathrm{dom}(\Phi_{e,s})$.

c.e. approximations

Proposition 2.7

An set A is c.e. iff it has a c.e. approximation.

Proof.

"$\Rightarrow$": Let $A = W_e$. Clearly, $\{W_{e,s}\}_{s\in\omega}$ is a c.e. approximation of W_e.

"$\Leftarrow$": Suppose $\{A_s\}_{s\in\omega}$ is a c.e. approximation of A. We define the function f which on x searches for the smallest s such that $A_s(x) = 1$, and then halts with output 1 (and otherwise searches indefinitely without ever halting). We have $f(x)\downarrow$ iff $x \in \bigcup_s A_s = A$. □

listing Theorem

Theorem 2.8

An nonempty set A is c.e. iff it is the range of a total computable function.

Proof.

"⇒": Let $A = W_e \neq \emptyset$. Choose any $a \in W_e$. Define

$$f(\langle s, x\rangle) = \begin{cases} x & \text{if } x \in W_{e,s+1} - W_{e,s}; \\ a & \text{otherwise.} \end{cases}$$

"⇐": Suppose $A = \mathrm{rng}(\Phi_e)$. We define the function g which on n searches for the smallest s such that there exists $x < s$ with $\Phi_{e,s}(x) \downarrow = n$, and then halts with output 1 (and otherwise searches indefinitely without ever halting). We have $g(n) \downarrow$ iff $\exists n \in \mathrm{rng}(\Phi_e) = A$. $\square$

halting problem

Definition 2.9

The *halting problem* K is the set

$$K = \{e \colon \Phi_e(e)\downarrow\} = \{e \colon e \in W_e\}.$$

Theorem 2.10

K is c.e. but not computable.

Proof.

By Enumeration Theorem, there is a partial computable function $\Psi(e,x)$ such that $\Psi(e,x) = \Phi_e(x)$. Let $f(x) = \Psi(x,x)$. Then f is also partial computable. Moreover, $K = \mathrm{dom}(f)$.

Suppose K is computable. Then the following function is total and computable.

$$f(x) = \begin{cases} \Phi_x(x) + 1 & \text{if } x \in K; \\ 0 & \text{otherwise.} \end{cases}$$

However, f cannot be computable because $f(x) \neq \Phi_x(x)$ for every x. $\square$

properties of c.e. sets

Corollary 2.11

The class of c.e. sets is not closed under complement.

halting problem K_0

Definition 2.12

Let $K_0 := \{\langle x, y\rangle : \Phi_x(y)\downarrow\} = \{\langle x, y\rangle : y \in W_x\}$.

Theorem 2.13

K_0 is c.e. but not computable.

Proof.

Let $f(\langle x, y\rangle) = \Psi(x, x)$. Then f is partial computable and $K_0 = \mathrm{dom}(f)$.

Suppose K_0 is computable. Then there exists computable function $f = \chi_{K_0}$. Let $g(x) = f(\langle x, x\rangle)$. Then g is computable and $g = \chi_K$. Thus, K is computable, contradicts to previous theorem. $\square$

m-reducibility and 1-reducibility

Definition 2.14

1. A is *many-one reducible* (m-reducible) to B (written $A \leq_{\mathrm{m}} B$) if there is a computable function f such that $f(A) \subseteq B$ and $f(\overline{A}) \subseteq \overline{B}$, i.e., $x \in A$ iff $f(x) \in B$.
2. A is *one-one reducible* (1-reducible) to B ($A \leq_1 B$) if $A \leq_{\mathrm{m}} B$ by a 1:1 computable function.

$K \leq_1 K_0$ via the function $f(x) = \langle x, x \rangle$.

Note that if $A \leq_{\mathrm{m}} B$ via f then $\overline{A} \leq_{\mathrm{m}} \overline{B}$ via f also.

Proposition 2.15

Given $A \leq_{\mathrm{m}} B$,

1. *if B is computable, then A is also computable;*
2. *if B is c.e., then A is also c.e..*

m-reducibility and 1-reducibility

Theorem 2.16

$K \leq_1 \mathrm{Tot} := \{e\colon \Phi_e \textit{ is a total function}\}$.

Proof.

- Define the function:

$$\Psi(x,y) = \begin{cases} 1 & \text{if } x \in K; \\ \uparrow & \text{otherwise.} \end{cases}$$

- By the s-m-n Theorem there is a 1:1 computable function $s(x,y)$ s.t. $\Phi_{s(x,y)}(z) = \Phi_x(y,z)$. Let e be in index of Ψ, and define $f(x) = \lambda x[s(e,x)]$. Then $\Phi_{f(x)}(y) = \Phi_e(x,y) = \Psi(x,y)$ and f is 1:1 because s is 1:1.
- Note that

$$\begin{array}{lllllll} x \in K & \Rightarrow & \Phi_{f(x)} = \lambda y[1] & \Rightarrow & \Phi_{f(x)} \text{ total} & \Rightarrow & f(x) \in \mathrm{Tot}; \\ x \notin K & \Rightarrow & \Phi_{f(x)} = \lambda y[\uparrow] & \Rightarrow & \Phi_{f(x)} \text{ not total} & \Rightarrow & f(x) \notin \mathrm{Tot}. \end{array}$$

□

index set

Definition 2.17

A set $A \in \mathbb{N}$ is an *index set* if for all $x, y \in \mathbb{N}$

$$(x \in A \wedge \Phi_x = \Phi_y) \Rightarrow y \in A$$

The following are index sets.

$$\begin{aligned}
K_1 &= \{x : W_x \neq \emptyset\}; \\
\mathrm{Fin} &= \{x : W_x \text{ is finite}\}; \\
\mathrm{Inf} &= \{x : W_x \text{ is infinite}\} = \mathbb{N} - \mathrm{Fin}; \\
\mathrm{Tot} &= \{x : \Phi_x \text{ is total}\} = \{x : W_x = \mathbb{N}\}; \\
\mathrm{Con} &= \{x : \Phi_x \text{ is total and constant}\}; \\
\mathrm{Cof} &= \{x : W_x \text{ is cofinite}\}; \\
\mathrm{Rec} &= \{x : W_x \text{ is computable (recursive)}\}.
\end{aligned}$$

Rice's Theorem

Theorem 2.18 (Rice's Theorem)

If A is a nontrivial index set, i.e., $A \neq \emptyset$, $A \neq \mathbb{N}$, then either $K \leq_1 A$ or $K \leq_1 \overline{A}$.

Proof.

Choose e_0 such that $\Phi_{e_0}(y) \uparrow$ for all y.
Suppose $e_0 \in \overline{A}$. Since $A \neq \emptyset$ we can choose $e_1 \in A$. Now $\Phi_{e_1} \neq \Phi_{e_0}$ because A is an index set.
By the s-m-n Theorem define a 1:1 computable function f such that

$$\Phi_{f(x)}(y) = \Psi_{(}x, y) = \begin{cases} \Phi_{e_1}(y) & \text{if } x \in K; \\ \uparrow & \text{otherwise.} \end{cases}$$

Now

$$\begin{aligned} x \in K \quad &\Rightarrow \quad \Phi_{f(x)} = \Phi_{e_1} \quad \Rightarrow \quad f(x) \in A, \\ x \in \overline{K} \quad &\Rightarrow \quad \Phi_{f(x)} = \Phi_{e_0} \quad \Rightarrow \quad f(x) \in \overline{A}. \end{aligned}$$

If $e_0 \in A$, then $K \leq_1 \overline{A}$ similarly. □

hardness and completeness

Let $\mathcal{E}$ denote the class of all c.e. sets.

Definition 2.19

For a reducibility r and a class C of sets, given a set A, we say

1. A is *r-hard for C* if $B \leq_r A$ for all $B \in C$.
2. A is *r-complete for C* if $A \in C$ and A is r-hard for C.
3. A is *r-hard/complete* if A is r-hard/complete for $\mathcal{E}$.

Proposition 2.20

K_0 is 1-complete.

Proof.

For any c.e. set W_e, $x \in W_e$ iff $f_e(x) := \langle e, x \rangle \in K_0$.

□

Proposition 2.21

K is 1-complete.

Proof.

By the s-m-n Theorem define a 1:1 computable function f such that

$$\Phi_{f(x)}(y) = \Psi_{(}x, y) = \begin{cases} 1 & \text{if } \Phi_e(x)\downarrow; \\ \uparrow & \text{otherwise.} \end{cases}$$

Now

$$\begin{array}{lllllll} x \in W_e & \Rightarrow & \forall y\, \Phi_{f(x)}(y) = 1 & \Rightarrow & \Phi_{f(x)}(f(x)) = 1 & \Rightarrow & f(x) \in K, \\ x \notin W_e & \Rightarrow & \forall y\, \Phi_{f(x)}(y)\uparrow & \Rightarrow & \Phi_{f(x)}(f(x))\uparrow & \Rightarrow & f(x) \notin K. \end{array}$$

$\square$