Foundations of Computability Theory (3)

Nan Fang

Institute of Software, Chinese Academy of Sciences

Peking University

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m-degrees and 1-degrees

Definition 3.1

A binary relation $R(x, y)$ is an *equivalence relation* if satisfies the following properties.

1. (reflexive) For all x, $R(x, x)$.
2. (symmetric) For all x, y, $R(x, y) \Leftrightarrow R(y, x)$.
3. (transitive) For all x, y, z, $R(x, y) \wedge R(y, z) \Rightarrow R(x, z)$.

Definition 3.2

For a reducibility r ($r = m, 1, \dots$), a set A is *r-equivalent* to a set B (written $A \equiv_r B$) if $A \leq_r B$ and $B \leq_r A$.

- $\equiv_{\mathrm{m}}$ and $\equiv_1$ are equivalence relations. There corresponding equivalence classed are called m-degrees and 1-degrees.
- For a set A, $\deg_{\mathrm{m}}(A) := \{B \colon A \equiv_{\mathrm{m}} B\}$, $\deg_1(A) := \{B \colon A \equiv_1 B\}$.

join

Definition 3.3

The *join* of two sets $A, B \subseteq \mathbb{N}$ is defined as

$$A \oplus B = \{2x : x \in A\} \cup \{2x + 1 : x \in B\}.$$

Proposition 3.4

For sets $A, B, C, D \subseteq \mathbb{N}$, the following properties hold.

1. $A \leq_{\mathrm{m}} A \oplus B$, $B \leq_{\mathrm{m}} A \oplus B$.
2. *If* $A \leq_{\mathrm{m}} C$ *and* $B \leq_{\mathrm{m}} C$, *then* $A \oplus B \leq_{\mathrm{m}} C$.
3. *If* $A \equiv_{\mathrm{m}} C$ *and* $B \equiv_{\mathrm{m}} D$, *then* $A \oplus B \equiv_{\mathrm{m}} C \oplus D$.
4. *If* A, B *are c.e. sets, then* $A \oplus B$ *is also a c.e. set.*

structure of m-degrees

Definition 3.5

A *partial order* $(P, \leq)$ is a binary relation with the following properties.

1. (reflexive) For all $x \in P$, $x \leq x$.
2. (antisymmetric) For all $x, y \in P$, $x \leq y \wedge y \leq x \Rightarrow x = y$.
3. (transitive) For all $x, y, z \in P$, $x \leq y \wedge y \leq z \Rightarrow x \leq z$.

Definition 3.6

A partial order $(P, \leq)$ is a

1. *upper semi-lattice* if any two element $a, b \in P$ have a least upper bound $(a \vee b)$;
2. *lower semi-lattice* if any two element $a, b \in P$ have a greatest lower bound $(a \wedge b)$;
3. *lattice* if it is both an upper semi-lattice and a lower semi-lattice.

- Let $\mathcal{D}_{\mathrm{m}}$ be the set of all m-degrees. Then $(\mathcal{D}_{\mathrm{m}}, \leq_{\mathrm{m}})$ is an upper semi-lattice.
- For $\mathbf{a} = \deg_{\mathrm{m}}(A)$ and $\mathbf{b} = \deg_{\mathrm{m}}(B)$, $\mathbf{a} \vee \mathbf{b} = \deg_{\mathrm{m}}(A \oplus B)$.

- An m-degree is c.e. if it contains a c.e. set.
- c.e. m-degrees are closed under join.

productive set and creative set

Definition 3.7

1. A set P is *productive* if there is a p.c. function $\Psi(x)$, called a *productive function* for P, such that
$$(\forall x)[W_x \subseteq P \;\Rightarrow\; [\Psi(x)\downarrow \;\&\; \Psi(x) \in P - W_x]]$$

2. A c.e. set C is *creative* if $\overline{C}$ is productive.

Proposition 3.8

K is creative.

Proof.

We show $\overline{K}$ is productive via the identity function $p(x) = x$.
Suppose $W_x \subseteq \overline{K}$.

- If $x \notin \overline{K}$, then $x \in K$, which implies $x \in W_x \subseteq \overline{K}$. $\lightning$
- If $x \in W_x \subseteq \overline{K}$, then $x \in K$. $\lightning$

Thus, $p(x) = x \in \overline{K} \setminus W_x$. $\square$

Theorem 3.9

If P is productive and $P \leq_m A$ then A is productive.

Proof.

Let $P \leq_m A$ via f, and let p be a productive function for P. Define

$$\Phi_{g(x)}(y) = \Psi(x,y) = \begin{cases} 1 & \text{if } f(y) \in W_x; \\ \uparrow & \text{o.w.} \end{cases}$$

Then $W_{g(x)} = f^{-1}(W_x)$. Now

$$W_x \subseteq A \;\Rightarrow\; W_{g(x)} = f^{-1}(W_x) \subseteq P \;\Rightarrow\; p(g(x)) \in P \setminus W_{g(x)} \;\Rightarrow\; f(p(g(x))) \in A \setminus W_x.$$

Then $f \circ p \circ g$ is a productive function for A. □

Theorem 3.10

Every productive set P has a total 1:1 computable productive function p.

Proof.

Let f be a productive function of P. By the s-m-n Theorem define a 1:1 computable function g such that

$$\Phi_{g(x)}(y) = \Psi(x,y) = \begin{cases} \Phi_x(y) & \text{if } f(x)\downarrow; \\ \uparrow & \text{o.w.} \end{cases}$$

Then

$$W_{g(x)} = \begin{cases} W_x & \text{if } f(x)\downarrow; \\ \emptyset & \text{o.w.} \end{cases}$$

Now for all x, if $f(x)\uparrow$, $W_{g(x)} = \emptyset \subset P$, and $f(g(x))\downarrow$. Let $q(x)$ be the value of $f(x)$ or $f(g(x))$ whichever converges faster. Then q is a total computable function.

If $W_x \subseteq P$, as $f(x)\downarrow$, then $W_{g(x)} = W_x \subseteq P$, which implies $f(g(x))\downarrow$ and $f(g(x)) \in P \setminus W_x$. Thus, no matter $q(x) = f(x)$ or $q(x) = f(g(x))$, we have $q(x)\downarrow$ and $q(x) \in P \setminus W_x$. Therefore, q is a productive function of P. □

Proof cont.

By the s-m-n Theorem define a 1:1 computable function h such that

$$W_{h(x)} = W_x \cup \{q(x)\}.$$

Since q is a productive function, then $W_x \subseteq P$ implies $W_{h(x)} \subseteq P$.

Now we define a function p as follows.

- Let $p(0) = q(0)$.
- Suppose $p(i)$ are defined for all $i < n$. Let $Y_n = \{q(n), q(h(n)), q(h^2(n)), \ldots, q(h^n(n))\}$.
- If there are $n+1$ many elements in Y_n, let $p(n) = \min\{Y_n \setminus \{p(0), p(1), \ldots, p(n-1)\}\}$.
- Otherwise, let $p(n) = \min\{\mathbb{N} \setminus \{p(0), p(1), \ldots, p(n-1)\}\}$.

Obviously, the function p is a 1:1 total computable function.

If $W_n \subseteq P$, then $Y_n \subseteq P$ and $Y_n \cap W_n = \emptyset$. Then by definition Y_n must have $n+1$ many elements. By definition, $p(n) \in Y_n$, which implies $p(n) \in P \setminus W_n$. □

Recursion Theorem with Parameters

Theorem 3.11 (Recursion Theorem with Parameters)

If $f(x, y)$ is a computable function, then there is a 1:1 computable function $n(y)$ such that $\Phi_{n(y)} = \Phi_{f(n(y),y)}$.

Proof.

- Define a partial computable function

$$\Phi_{d(x,y)}(z) = \Psi(x, y, z) = \begin{cases} \Phi_{\Phi_x(x,y)}(z) & \text{if } \Phi_x(x, y)\downarrow; \\ \uparrow & \text{o.w.} \end{cases}$$

- Choose v such that $\Phi_v(x, y) = f(d(x, y), y)$.
- Then $n(y) = \lambda y[d(v, y)]$ is a fixed point, because

$$\Phi_{d(v,y)} = \Phi_{\Phi_v(v,y)} = \Phi_{f(d(v,y),y)}.$$

The first equality follows from the definition of d and the second from the definition of v.

□

Theorem 3.12

If P is productive then $\overline{K} \leq_1 P$.

Proof.

Let p be a total 1:1 computable productive function for P. Define the computable function f by

$$\Phi_{f(x,y)}(z) = \Psi(x,y,z) = \begin{cases} 1 & \text{if } y \in K \& z = p(x); \\ \uparrow & \text{otherwise.} \end{cases}$$

By the Recursion Theorem with Parameters there is a 1:1 computable function $n(y)$ such that $\Phi_{n(y)} = \Phi_{f(n(y),y)}$. Thus

$$W_{n(y)} = W_{f(n(y),y)} = \begin{cases} \{p(n(y))\} & \text{if } y \in K; \\ \emptyset & \text{otherwise.} \end{cases}$$

Now

$$y \in \overline{K} \quad \Rightarrow \quad W_{n(y)} = \emptyset \quad \Rightarrow \quad W_{n(y)} \subseteq P \quad \Rightarrow \quad p(n(y)) \in P.$$

$$y \in K \quad \Rightarrow \quad W_{n(y)} = \{p(n(y))\} \quad \Rightarrow \quad W_{n(y)} \nsubseteq P \quad \Rightarrow \quad p(n(y)) \in \overline{P}$$

□

Corollary 3.13

The following are equivalent:

(i) *P is productive;*

(ii) $\overline{K} \leq_1 P$;

(iii) $\overline{K} \leq_{\mathrm{m}} P$.

Corollary 3.14

The following are equivalent:

(i) *C is creative;*

(ii) *C is 1-complete;*

(iii) *C is m-complete.*

computable permutations and isomorphisms

Definition 3.15

1. A *computable permutation* is a bijection from $\mathbb{N}$ to $\mathbb{N}$.
2. A is *computably isomorphic* to B (written $A \equiv B$) if there is a computable permutation p such that $p(A) = B$.

Theorem 3.16 (Myhill Isomorphism Theorem)

$A \equiv B$ iff $A \equiv_1 B$.

Proof of Myhill Isomorphism Theorem

"$\Rightarrow$": Trivial.
"$\Leftarrow$": Let $A \leq_1 B$ via f and $B \leq_1 A$ via g. Therefore,

$$(\forall x)(\forall y)[x \in A \iff f(x) \in B \quad \& \quad y \in B \iff g(y) \in A]. \tag{1}$$

We define a computable permutation h by stages $s \in \omega$ so that $h(A) = B$. Suppose that by the end of stage $2s$ we have finite sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ and a 1:1 map h such that for all $1 \leq i \leq n$

$$h(x_i) = y_i \quad \& \quad x_i \in A \iff h(x_i) \in B. \tag{2}$$

Stage 2s + 1. We shall define $h(s)$ if it is not already defined.

- Compute $f(s) = t_1$. If $t_1 \notin Y$ define $h(s) = t_1$. If $t_1 \in Y$, say $t_1 = y_i$, then take $x_i = h^{-1}(t_1)$. Note that $x_i \in A$ iff $s \in A$ by (1) and (2).
- Compute $f(x_i) = t_2$. If $t_2 \notin Y$ define $h(s) = t_2$. Otherwise, $t_2 = y_j$ for some j. Take $x_j = h^{-1}(y_j)$ and note that $x_j \in A$ iff $s \in A$ as before.
- Compute $f(x_j) = t_3$. Continue in this fashion until a new element $z \notin Y$ is found and define $h(s) = z$.
- Note that $X \cup \{s\}$ has $n + 1$ elements but Y has only n elements, so the procedure must terminate.

Stage 2s + 2. Find the value of $h^{-1}(s)$ in similar fashion using h^{-1} and g in place of h and f. $\square$