Foundations of Computability Theory (6)

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Σ^0_1 normal form for c.e. sets

Definition 6.1

1. A set A is a *projection* of some relation $R \subseteq \omega \times \omega$ if
$$A = \{x : (\exists y)\ R(x, y)\}$$
(i.e., geometrically A is the projection of the two-dimensional relation R onto the x-axis).
2. A set A is in Σ^0_1*-form* $(A \in \Sigma^0_1)$, if A is the projection of some computable relation $R \subseteq \omega \times \omega$.

Theorem 6.2 (Normal Form Theorem for C.E. Sets)

A set A is c.e. iff A is Σ^0_1.

Proof.

"$\Rightarrow$:" If A is c.e. then $A = W_e := \mathrm{dom}(\Phi_e)$ for some e. Hence,
$$x \in W_e \Leftrightarrow (\exists s)[x \in W_{e,s}].$$
Clearly, the relation $\{\langle e, x, s\rangle : x \in W_{e,s}\}$ is computable.
"$\Leftarrow$:" Let $A = \{x : (\exists y)R(x, y)\}$, where R is computable. Then $A = \mathrm{dom}(\Psi)$ where
$$\Psi(x) = (\mu y)\ R(x, y).$$

quantifier contraction

Theorem 6.3 (Quantifier Contraction Theorem)

A is Σ_1^0 iff there is an $n \in \omega$ and a computable relation $R \subseteq \omega^{n+1}$ such that

$$A = \{x : (\exists y_1)(\exists y_2)\cdots(\exists y_n)\ R(x, y_1, \ldots, y_n)\}.$$

Proof.

Define the computable relation $S \subseteq \omega^2$ by

$$S(x,z) \quad \text{if and only if} \quad R(x, (z)_1, (z)_2, \ldots, (z)_n),$$

where $z = p_0^{(z)_1} p_1^{(z)_2} \cdots p_n^{(z)_n}$ is the prime decomposition. Then

$$\begin{aligned} (\exists z)S(x,z) \quad &\Leftrightarrow \quad (\exists z)\ R(x, (z)_1, (z)_2, \ldots, (z)_n) \\ &\Leftrightarrow \quad (\exists y_1)(\exists y_2)\cdots(\exists y_n)\ R(x, y_1, y_2, \ldots, y_n) \end{aligned}$$

$\square$

Corollary 6.4

The projection of a c.e. relation is c.e.

the arithmetical hierarchy

Note that here Σ_n, Π_n, and Δ_n are abbreviated from Σ^0_n, Π^0_n, and Δ^0_n.

Definition 6.5

1. A set B is Σ_0 (Π_0, Δ_0) iff B is computable.
2. For $n \geq 1$, B is Σ_n (written $B \in \Sigma_n$) if there is a computable relation $R(x, y_1, y_2, \ldots, y_n)$ such that
$$x \in B \Leftrightarrow (\exists y_1)(\forall y_2)(\exists y_3) \cdots (Q y_n)\ R(x, y_1, y_2, \ldots, y_n),$$
where Q is $\exists$ for n odd, and $\forall$ for n even.
3. Likewise, B is Π_n $(B \in \Pi_n)$ if
$$x \in B \Leftrightarrow (\forall y_1)(\exists y_2)(\forall y_3) \cdots (Q y_n)\ R(x, y_1, y_2, \ldots, y_n),$$
where Q is $\exists$ or $\forall$ according to whether n is even or odd.
4. B is Δ_n $(B \in \Delta_n)$ if $B \in \Sigma_n$ and $B \in \Pi_n$.
5. B is *arithmetical* if $B \in \bigcup_n (\Sigma^0_n \cup \Pi^0_n)$.

Definition 6.6

Fix a set A. If we replace everywhere "computable" in Definition 6.5 by "A-computable" then we have the definition of B being Σ_n in A (written $B \in \Sigma^A_n$), B being Π_n in A $(B \in \Pi^A_n)$, $B \in \Delta^A_n$, and B being arithmetical in A.

Theorem 6.7

(i) $A \in \Sigma_n \Leftrightarrow \overline{A} \in \Pi_n$;

(ii) $A \in \Sigma_n(\textit{ or } \Pi_n) \Rightarrow (\forall m > n)[A \in \Sigma_m \cap \Pi_m]$;

(iii) $A, B \in \Sigma_n(\Pi_n) \Rightarrow A \cup B, A \cap B \in \Sigma_n(\Pi_n)$;

(iv) $[R \in \Sigma_n \ \wedge \ n > 0 \ \wedge \ A = \{x\colon (\exists y)\ R(x,y)\}] \Rightarrow A \in \Sigma_n$;

Proof.

(i) If $A = \{x : (\exists y_1)(\forall y_2) \cdots R(x, \vec{y})\}$, then $\overline{A} = \{x : (\forall y_1)(\exists y_2) \cdots \neg R(x, \vec{y})\}$.

(ii) For example, if $A = \{x : (\exists y_1)(\forall y_2) R(x, y_1, y_2)\}$, then
$A = \{x : (\exists y_1)(\forall y_2)(\exists y_3)[R(x, y_1, y_2) \wedge y_3 = y_3]\}$.

(iii) Let $A = \{x : (\exists y_1)(\forall y_2) \cdots R(x, \vec{y})\}$, and $B = \{x : (\exists z_1)(\forall z_2) \cdots S(x, \vec{z})\}$. Then

$$\begin{aligned} x \in A \cup B &\Leftrightarrow (\exists y_1)(\forall y_2) \cdots R(x, \vec{y}) \vee (\exists z_1)(\forall z_2) \cdots S(x, \vec{z}), \\ &\Leftrightarrow (\exists y_1)(\exists z_1)(\forall y_2)(\forall z_2) \cdots [R(x, \vec{y}) \vee S(x, \vec{z})], \\ &\Leftrightarrow (\exists u_1)(\forall u_2) \cdots [R(x, (u_1)_0, (u_2)_0 \ldots) \vee S(x, (u_1)_1, (u_2)_1 \ldots)], \end{aligned}$$

where $u_i = \langle (u_i)_0, (u_i)_1 \rangle$ is the pairing function. Likewise, this holds for $A \cap B$.

(iv) Immediately by quantifier contraction, as in (iii).

$\square$

the arithmetical hierarchy

Theorem 6.7

(v) $[B \leq_{\mathrm{m}} A \ \wedge \ A \in \Sigma_n] \Rightarrow B \in \Sigma_n$;

(vi) If $R \in \Sigma_n(\Pi_n)$, then $A, B \in \Sigma_n(\Pi_n)$ for A and B defined by

$$\begin{aligned} \langle x, y\rangle \in A \quad &\Leftrightarrow \quad (\forall z < y) R(x, y, z), \\ \langle x, y\rangle \in B \quad &\Leftrightarrow \quad (\exists z < y) R(x, y, z). \end{aligned}$$

Proof.

(v) Let $A = \{x : (\exists y_1)(\forall y_2) \cdots R(x, \vec{y})\}$ and $B \leq_m A$ via f. Then $B = \{x : (\exists y_1)(\forall y_2) \cdots R(f(x), \vec{y})\}$.

(vi) The proof is by induction on n. If $n = 0$, then A and B are clearly computable. Fix $n > 0$. Suppose $R \in \Sigma_n$ and assume (vi) for all $m < n$. Then $B \in \Sigma_n$ by (iv). Now there exists $S \in \Pi_{n-1}$ such that

$$\begin{aligned} \langle x, y\rangle \in A \quad &\Leftrightarrow \quad (\forall z < y)\ R(x, y, z) \quad \Leftrightarrow \quad (\forall z < y)(\exists u)\ S(x, y, z, u) \\ &\Leftrightarrow \quad (\exists \sigma)(\forall z < y)\ S(x, y, z, \sigma(z)), \end{aligned}$$

where σ ranges over $\omega^{<\omega}$. Now $(\forall z < y) S \in \Pi_{n-1}$ by the inductive hypothesis, so $A \in \Sigma_n$. The case $R \in \Pi_n$ follows from the case $R \in \Sigma_n$ by (i). $\square$

examples: index sets

Proposition 6.8

Fin $\in \Sigma_2$.

Proof.

$$x \in \text{Fin} \quad \Leftrightarrow \quad W_x \text{ is finite} \quad \Leftrightarrow \quad (\exists s)(\forall t)[t \leq s \ \vee \ W_{x,t} = W_{x,s}].$$

The bracketed relation of x, s, t is clearly computable. $\square$

Proposition 6.9

Cof $\in \Sigma_3$.

Proof.

$$\begin{aligned} x \in \text{Cof} \quad &\Leftrightarrow \quad \overline{W}_x \text{ is finite} \quad \Leftrightarrow \quad (\exists y)(\forall z)[z \leq y \ \vee \ z \in W_x] \\ &\Leftrightarrow \quad (\exists y)(\forall z)(\exists s)[z \leq y \ \vee \ z \in W_{x,s}]. \end{aligned}$$

$\square$

examples: index sets

Proposition 6.10

$\{\langle x, y\rangle : W_x \subseteq W_y\} \in \Pi_2$.

Proof.

$$\begin{aligned} W_x \subseteq W_y \quad &\Leftrightarrow \quad (\forall z)[z \in W_x \Rightarrow z \in W_y] \quad \Leftrightarrow \quad (\forall z)[z \notin W_x \vee z \in W_y] \\ &\Leftrightarrow \quad (\forall z)[\forall \vee \exists] \quad \Leftrightarrow \quad \forall\forall\exists[\dots] \quad \Leftrightarrow \quad \forall\exists[\dots]. \end{aligned}$$

□

Corollary 6.11

(i) $\{\langle x, y\rangle : W_x = W_y\} \in \Pi_2$.

(ii) $\mathrm{Tot} = \{y : W_y = \omega\} \in \Pi_2$.

Proof.

(i) follows by Proposition 6.10 and Theorem 6.7(iii), and (ii) follows from Proposition 6.10 with $W_x = \omega$. □

Σ_n-completeness, Post's theorem and hierarchy theorem

Definition 6.12

A set A is Σ_n-*complete* (Π_n-*complete*) if $A \in \Sigma_n$ (Π_n) and $B \leq_1 A$ for every $B \in \Sigma_n$ (Π_n).

Note that A is Σ_1-complete iff A is 1-complete. Hence, K is Σ_1-complete and $\overline{K}$ is Π_1-complete.

Theorem 6.13 (Post's Theorem)

For every $n \geq 0$,

(i) $B \in \Sigma_{n+1} \quad \Leftrightarrow \quad$ *B is c.e. in some Π_n set*
$\quad\quad \Leftrightarrow \quad$ *B is c.e. in some Σ_n set (by Jump Theorem (viii)).*

(ii) $\emptyset^{(n+1)}$ *is Σ_{n+1}-complete;*

(iii) $B \in \Sigma_{n+1} \Leftrightarrow$ *B is c.e. in $\emptyset^{(n)}$;*

(iv) $B \in \Delta_{n+1} \Leftrightarrow B \leq_{\mathrm{T}} \emptyset^{(n)}$.

Corollary 6.14 (Hierarchy Theorem)

$(\forall n > 0)[\Delta_n \subsetneq \Sigma_n \;\&\; \Delta_n \subsetneq \Pi_n]$. *Clearly, $\Delta_n \subseteq \Sigma_n$. The content here is that $\Sigma_n \not\subseteq \Delta_n$.*

Proof.

$\emptyset^{(n)} \in \Sigma_n - \Delta_n$, by Post's Theorem (ii) and (iv), and the Jump Theorem (ii). Likewise, $\overline{\emptyset^{(n)}} \in \Pi_n - \Delta_n$. $\square$

proof of Post's theorem

Theorem 6.13 (Post's Theorem)

(i) $B \in \Sigma_{n+1} \quad \Leftrightarrow \quad B$ is c.e. in some Π_n set
$\quad \Leftrightarrow \quad B$ is c.e. in some Σ_n set (by Jump Theorem (viii)).

Proof.

(i) "$\Rightarrow$:" Let $B \in \Sigma_{n+1}$. Then $x \in B \Leftrightarrow (\exists y)R(x,y)$ for some $R \in \Pi_n$. Hence B is Σ_1 in R and therefore c.e. in R by relativization of Theorem 6.2.
"$\Leftarrow$:" Suppose B is c.e. in some Π_n set C. Then for some e,

$$x \in B \quad \Leftrightarrow \quad x \in W_e^C \quad \Leftrightarrow \quad (\exists s)(\exists \sigma)[\sigma \prec C \ \& \ x \in W_{e,s}^{\sigma}].$$

Clearly, $x \in W_{e,s}^{\sigma}$ is computable. Now it suffices to show that $\sigma \prec C$ is Σ_{n+1}. Note that

$$\sigma \prec C \Leftrightarrow (\forall y < |\sigma|)[\sigma(y) = C(y)] \Leftrightarrow (\forall y < |\sigma|)[(\sigma(y) = 1 \land y \in C) \lor (\sigma(y) = 0 \land y \notin C)]$$

$$\Leftrightarrow (\forall y < |\sigma|)[(\Pi_n \lor \Sigma_n)] (\text{because } C \in \Pi_n)$$

□

proof of Post's theorem

Theorem 6.13 (Post's Theorem)

(ii) $\emptyset^{(n+1)}$ is Σ_{n+1}-complete;

(iii) $B \in \Sigma_{n+1} \Leftrightarrow B$ is c.e. in $\emptyset^{(n)}$;

(iv) $B \in \Delta_{n+1} \Leftrightarrow B \leq_{\mathrm{T}} \emptyset^{(n)}$.

Proof.

(ii) This is proved by induction on n and is clear for $n = 0$. Fix $n \geq 1$ and assume $\emptyset^{(n)}$ is Σ_n-complete. As $\emptyset^{(n+1)}$ is $\emptyset^{(n)}$-c.e., by (i) it is Σ_{n+1}. Now

$$\begin{aligned} B \in \Sigma_{n+1} \quad &\Leftrightarrow \quad B \text{ is c.e. in some } \Sigma_n \text{ set} \quad \text{(by (i))} \\ &\Leftrightarrow \quad B \text{ is c.e. in } \emptyset^{(n)} \quad \text{(by inductive hypothesis and Jump Theorem (v))} \\ &\Leftrightarrow \quad B \leq_1 \emptyset^{(n+1)} \quad \text{(by Jump Theorem (iii)).} \end{aligned}$$

Hence, $\emptyset^{(n+1)}$ is Σ_{n+1}-complete.

(iii) Already shown in the proof of (ii).

(iv) $B \in \Delta_{n+1} \quad \Leftrightarrow \quad B, \overline{B} \in \Sigma_{n+1} \quad \Leftrightarrow \quad B, \overline{B}$ are c.e. in $\emptyset^{(n)}$ (by (iii)) $\quad \Leftrightarrow \quad B \leq_{\mathrm{T}} \emptyset^{(n)}$.

□

m-reducibility for pairs

Definition 6.15

Let A, B and C, D be two pairs of disjoint sets.

1. $(A, B) \leq_m (C, D)$ if there is a computable function f such that $f(A) \subseteq C$, $f(B) \subseteq D$, and $f(\overline{A \cup B}) \subseteq \overline{C \cup D}$. We write "$\leq_1$" if f is 1:1.
2. For $n \geq 1$ define $(\Sigma_n, \Pi_n) \leq_m (C, D)$ if $(A, \overline{A}) \leq_m (C, D)$ for some Σ_n-complete set A, and similarly for $\leq_1$ in place of $\leq_m$. In this case we also write $\Sigma_n \leq_m C$ and $\Pi_n \leq_m D$. (it can be shown that it makes no difference whether we write "$\leq_m$" or "$\leq_1$" here.)

classifying Σ_2 and Π_2 sets: Fin, Inf, and Tot

Theorem 6.16

$(\Sigma_2, \Pi_2) \leq_1 (\mathrm{Fin}, \mathrm{Tot})$. *Therefore,* Fin *is* Σ_2*-complete,* Inf *and* Tot *are* Π_2*-complete, and* Inf $\equiv_1$ Tot. *Hence,* Inf *and* Tot *are computably isomorphic.*

Proof.

By Proposition 6.8 and Corollary 6.11, Fin $\in \Sigma_2$ (so Inf $\in \Pi_2$) and Tot $\in \Pi_2$. Fix $A \in \Sigma_2$. Therefore, $\overline{A} \in \Pi_2$, and there is a computable relation R such that

$$x \in \overline{A} \Leftrightarrow (\forall y)(\exists z)R(x, y, z).$$

Using the *s-m-n* Theorem, define a $1 : 1$ computable function f by

$$\Phi_{f(x)}(u) = \begin{cases} 0 & \text{if } (\forall y \leq u)(\exists z)R(x, y, z); \\ \uparrow & \text{otherwise.} \end{cases}$$

Now

$$\begin{array}{lllll} x \in \overline{A} & \Rightarrow & W_{f(x)} = \omega & \Rightarrow & f(x) \in \mathrm{Tot}, \\ x \in A & \Rightarrow & W_{f(x)} \text{ is finite} & \Rightarrow & f(x) \in \mathrm{Fin}. \end{array}$$

□