Foundations of Computability Theory (7)

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principal function

Definition 7.1

Given a set $A \subseteq \omega$, let $\{a_s\}_{s\in\omega}$ be a list of the elements of A in magnitude order, that is, $A = \{a_0 < a_1 < a_2 < \dots\}$. The *principal function* of A is defined as

$$p_A(x) = a_x.$$

Classifying Cof as Σ_3-Complete

Theorem 7.2

Cof *is* Σ_3*-complete.*

Proof.

Fix $A \in \Sigma_3$. Now for some relation $R \in \Pi_2$, $x \in A \Leftrightarrow (\exists y)R(x, y)$. Since $R \in \Pi_2$ and Inf is Π_2-complete, there is a computable function g such that $R(x, y)$ iff $W_{g(x,y)}$ is infinite. Therefore,

$$x \in A \Leftrightarrow (\exists y)[W_{g(x,y)} \text{ is infinite}]. \tag{1}$$

We shall define a c.e. set B_x uniformly in x such that $x \in A$ iff B_x is cofinite. Fix x. We enumerate $B_x = \bigcup_{s\in\omega} B_{x,s}$ by stages s in the following computable construction.

Stage $s = 0$. Set $B_{x,0} = \emptyset$.
Stage $s + 1$. For each $y \leq s$ such that $W_{g(x,y),s} \neq W_{g(x,y),s+1}$, enumerate $b_{y,s} = p_{\overline{B_{x,s}}}(y)$ in $B_{x,s+1}$. If no such y exists, define $B_{x,s+1} = B_{x,s}$. This ends the construction.

Case 1. $x \in A$. By (1), choose the least y such that $W_{g(x,y)}$ is infinite. Now there are infinitely many stages s such that $W_{g(x,y),s} \neq W_{g(x,y),s+1}$. Therefore, $\lim_s b_{y,s} = \infty$, and $|\overline{B}| \leq y$.
Case 2. $x \notin A$. By induction, fix y, and choose s such that $W_{g(x,y),s} = W_{g(x,y),s+1}$ and for all $z < y$ such that $b_{z,s} = b_z$. Now $b_{y,s}$ never goes to $B_{x,s+1}$ after s. Hence, $\lim_s b_{y,s} < \infty$ and comes to rest on $\overline{B}$, which is therefore infinite. □

relativizing hierarchy results

Theorem 7.3 (Relativized Post's Theorem)

For every $n \geq 0$ and $A \subseteq \omega$,

(i) $B \in \Sigma^A_{n+1} \quad \Leftrightarrow \quad$ *B is c.e. in some Π^A_n set* $\quad \Leftrightarrow \quad$ *B is c.e. in some Σ^A_n set.*

(ii) $A^{(n+1)}$ *is Σ^A_{n+1}-complete;*

(iii) $B \in \Sigma^A_{n+1} \quad \Leftrightarrow \quad$ *B is c.e. in $A^{(n)}$;*

(iv) $B \in \Delta^A_{n+1} \quad \Leftrightarrow \quad B \leq_{\mathrm{T}} A^{(n)}$.

- Define Fin^A, Tot^A, and Cof^A as before but with W^A_e in place of W_e, and define Rec^A as the set of e's such that W^A_e is A-computable.
- By relativizing previous results to A we get: Fin^A is Σ^A_2-complete, Tot^A is Π^A_2-complete, and Cof^A and Rec^A are Σ^A_3-complete.
- If $\mathbf{a} = \deg(A)$, then $\mathbf{a}' = \deg(A')$, $\mathbf{a}'' = \deg(\mathrm{Fin}^A)$, and $\mathbf{a}''' = \deg(\mathrm{Cof}^A) = \deg(\mathrm{Rec}^A)$.

Lemma 7.4 (Relativized Shoenfield limit lemma)

$$B \leq_{\mathrm{T}} A' \quad \Leftrightarrow \quad (\exists f(x,s) \leq_{\mathrm{T}} A)(\forall x)\ [B(x) = \lim_s f(x,s)].$$

High Theorem

Theorem 7.5 (High Theorem)

For $A \subseteq \omega$, the following are equivalent:

(i) A *is* high *(i.e., $\emptyset'' \leq_T A'$);*

(ii) $\Sigma_2 \subseteq \Delta_2^A$;

(iii) $\Sigma_2 \subseteq \Pi_2^A$;

(iv) $\overline{\emptyset''} \leq_1 A''$ *(i.e.,* $\mathrm{Tot} \leq_1 \mathrm{Fin}^A$*).*

Proof.

$$
\begin{aligned}
A \text{ is high} \quad &\Leftrightarrow \quad \emptyset'' \leq_T A' \\
&\Leftrightarrow \quad \emptyset'' \in \Delta_2^A \quad \text{(by Post's Theorem 7.3)} \\
&\Leftrightarrow \quad \Sigma_2 \subseteq \Delta_2^A \quad \text{(because } \emptyset'' \text{ is } \Sigma_2\text{-complete)} \\
&\Leftrightarrow \quad \Sigma_2 \subseteq \Pi_2^A \quad \text{(because } \Sigma_2 \subseteq \Sigma_2^A \text{ trivially)} \\
&\Leftrightarrow \quad \emptyset'' \leq_1 \overline{A''} \quad \text{(because } \emptyset'' \in \Sigma_2\text{-complete and } \overline{A''} \text{ is } \Pi_2^A\text{-complete)} \\
&\Leftrightarrow \quad \overline{\emptyset''} \leq_1 A''.
\end{aligned}
$$

□

Low Theorem

Theorem 7.6 (Low Theorem)

For $A \leq_{\mathrm{T}} \emptyset'$, the following are equivalent:

(i) A *is* low *(i.e., $A' \leq_{\mathrm{T}} \emptyset'$);*

(ii) $\Sigma_1^A \subseteq \Delta_2$;

(iii) $\Sigma_1^A \subseteq \Pi_2$;

(iv) $A' \leq_1 \overline{\emptyset''} \equiv \mathrm{Tot}$.

Proof.

$$
\begin{aligned}
A \text{ is low} \quad &\Leftrightarrow \quad A' \leq_{\mathrm{T}} \emptyset' \\
&\Leftrightarrow \quad A' \in \Delta_2 \quad \text{(by Post's Theorem)} \\
&\Leftrightarrow \quad \Sigma_1^A \subseteq \Delta_2 \quad \text{(because } A' \text{ is } \Sigma_1^A\text{-complete)} \\
&\Leftrightarrow \quad \Sigma_1^A \subseteq \Pi_2 \quad \text{(because } A \leq_{\mathrm{T}} \emptyset'\text{, then } \Sigma_1^A \subseteq \Sigma_1^{\emptyset'} = \Sigma_2) \\
&\Leftrightarrow \quad A' \leq_1 \overline{\emptyset''} \quad \text{(because } A' \in \Sigma_1^A\text{-complete and } \overline{\emptyset''} \text{ is } \Pi_2\text{-complete).}
\end{aligned}
$$

□

- Similar results can be obtained for low$_n$ and high$_n$ sets.

domination and escaping domination

- $(\exists^\infty x)R(x)$ abbreviates $(\forall y)(\exists z > y)R(z)$, i.e., "there exist infinitely many x s.t. $R(x)$."
- $(\forall^\infty x)R(x)$ abbreviates $(\exists y)(\forall z > y)R(z)$, i.e., "for almost every x, $R(x)$."

These quantifiers are dual because $(\forall^\infty x)R(x)$ iff $\neg(\exists^\infty x)\neg R(x)$.

Definition 7.7

- A function g *dominates* f, denoted by $f <^* g$, if

$$(\forall^\infty x)[f(x) < g(x)].$$

  A partial function $\theta(x)$ *dominates* a partial function $\psi(x)$ if

$$(\forall^\infty x)[\psi(x)\downarrow \quad \Rightarrow \quad \psi(x) < \theta(x)\downarrow].$$

- A function f *escapes* (*domination* by) g if $f \not<^* g$, i.e., if

$$(\exists^\infty x)[g(x) \leq f(x)].$$

- A function g *majorizes* f, denoted by $f < g$, if

$$(\forall x)[f(x) < g(x)].$$

domination and escaping domination

Definition 7.8

- Functions f and g are *almost equal*, denoted by $f =^* g$, if

$$(\forall^\infty x)[g(x) = f(x)].$$

- A class C of functions is *closed under finite differences* if

$$[g \in C \quad \& \quad g =^* h] \quad \Rightarrow \quad h \in C.$$

Proposition 7.9

Let C be a class of functions closed under finite differences, such as the computable functions or the A-computable functions for some A. Then for every f,

$$(\exists g \in C)\ [g >^* f] \quad \Leftrightarrow \quad (\exists h \in C)\ [h > f].$$

By this proposition, given such a C and $g \in C$ with $g >^* f$, we shall assume that $g > f$. In particular, if we have a computable $g >^* f$ then we shall assume we have computable $g > f$.

domination properties

Definition 7.10

Let $\{A_s\}_{s\in\omega}$ be a c.e. approximation of a c.e. set A.

1. The *stage function* is
$$\theta_A(x) = \begin{cases} (\mu s)[x \in A_s] & \text{if } x \in A; \\ \uparrow & \text{otherwise.} \end{cases}$$

2. The *least modulus* is
$$m_A(x) = (\mu s)[A_s \upharpoonright x = A \upharpoonright x].$$

Note that $\theta_A(x)$ is *partial* but *partial computable*, while $m_A(x)$ is *total* but only *A-computable*.

domination properties

Theorem 7.11 (Domination Properties)

Let $\{A_s\}_{s\in\omega}$ be a c.e. approximation of a c.e. set A and f a total function.

(i) *If f dominates $\theta_A(x)$ then $A \leq_T f$.*

(ii) *For any $D \leq_T \emptyset'$,*

$$D \equiv_T \emptyset' \iff (\exists f \leq_T D)[f \text{ dominates every partial computable function}].$$

(iii) *If f dominates $m_A(x)$ then $A \leq_T f$.*

(iv) *If $\{B_s\}_{s\in\omega}$ is a c.e. approximation of a c.e. set B and $m_A(x)$ dominates the least modulus $m_B(x)$, then $B \leq_T A$.*

Proof.

(i) $(\forall^\infty x)[x \in A \iff x \in A_{f(x)}]$.

(ii) "$\Leftarrow$": By (i) because f dominates $\theta_K(x)$.
"$\Rightarrow$": Build $f \leq_T \emptyset'$ by using $\emptyset'$ to determine for a given input x which $\Phi_e(x)$ converge for $e \leq x$. Then define $f(x)$ to exceed all these values.

(iii) $(\forall^\infty x)[x \in A \iff x \in A_{f(x)}]$.

(iv) $(\forall^\infty x)[x \in B \iff x \in B_{m_A(x)}]$.

□

Martin's high domination theorem

Definition 7.12

A function f is *dominant* if f dominates every (total) computable function; an infinite set A is *dominant* if its principal function $p_A(n)$ is dominant.

Theorem 7.13 (High Domination Theorem, Martin 1966)

A set A is high *($\emptyset'' \leq_{\mathrm{T}} A'$) iff there is a dominant function $f \leq_{\mathrm{T}} A$.*

- This theorem is similar to part (ii) of Theorem 7.11 but now A is only high, not Turing complete.
- Our proof idea is similar, but more subtle because without Turing completeness of A, we cannot simply use A to determine which computable functions are total.
- However, we could still *guess* which computable functions are total and make sure our guesses are eventually correct by the following observation:

As $\mathrm{Tot} \equiv_{\mathrm{T}} \emptyset''$, by the Limit Lemma relativized to A, we have

$$\emptyset'' \leq_{\mathrm{T}} A' \quad \Leftrightarrow \quad \mathrm{Tot} \leq_{\mathrm{T}} A' \quad \Leftrightarrow \quad (\exists\, g(e,s) \leq_{\mathrm{T}} A)(\forall e)\ [\lim_s g(e,s) = \mathrm{Tot}(e)]$$

Proof of Martin's High Domination Theorem

"$\Rightarrow$": Assume $\emptyset'' \leq_T A'$. Given $g(e,s)$ as above we define a dominant function $f \leq_T A$.
At stage s: For all $e \leq s$ define $t(e)$ and $f(s)$ as follows:

$$t(e) = (\mu t > s)[g(e,t) = 0 \ \vee \ \Phi_{e,t}(s)\downarrow], \qquad f(s) = \max\{t(e) : e \leq s\}.$$

Note that $t(e)$ is well-defined because if $\Phi_e \uparrow$, then Φ_e is not total, and then $\lim_t g(e,t) = 0$. Now if Φ_e is total, then $\lim_t g(e,t) = 1$. Thus, for all sufficiently large $s \geq e$, in the definition of $t(e)$ only the second disjunct applies. Then we have $f(s) \geq t(e) > \Phi_e(s)$. (Recall that by our convention, if $\Phi_{e,t}(x)\downarrow = y$ then $e, x, y < t$.) Therefore, $f(s)$ dominates the total function Φ_e.

"$\Leftarrow$": Assume $f \leq_T A$ is dominant. Define an A-computable function $g(e,s)$ such that $\lim_s g(e,s) = \mathrm{Tot}(e)$ as follows:

$$g(e,s) = \begin{cases} 1 & \text{if } (\forall z \leq s)[\Phi_{e,f(s)}(z)\downarrow]; \\ 0 & \text{otherwise.} \end{cases}$$

Now if Φ_e is total, then so is $\theta_e(x) = (\mu s)(\forall z \leq x)[\Phi_{e,s}(z)\downarrow]$. Thus, $f(x)$ dominates $\theta_e(x)$.
Therefore, $g(e,s) = 1$ for almost every s.
If Φ_e is not total, then there exists x such that $\Phi_e(x)\uparrow$. And then by definition $g(e,s) = 0$ for all $s \geq x$. $\square$

nonlow$_2$ escape theorem

Let's restate Martin's High Domination Theorem as follows.

$$\emptyset'' \leq_T A' \Leftrightarrow \exists f \leq_T A \text{ such that f dominates every } (\emptyset\text{-})\text{computable function.}$$

We can relativize it by replacing both $\emptyset$ and A with B and $\emptyset'$ respectively

$$B'' \leq_T \emptyset'' \Leftrightarrow \exists f \leq_T \emptyset' \text{ such that f dominates every } B\text{-computable function.}$$

Note that $\emptyset \leq_T A$ holds implicitly in the original theorem, we need to add that $B \leq_T \emptyset'$ in the relativized version. Then we get the following theorem.

Theorem 7.14 (Relativized Domination Theorem, Martin 1966)

Fix $A \leq_T \emptyset'$. Then $A'' \leq_T \emptyset''$ (i.e., A is low$_2$*) if and only if there is a function $g \leq_T \emptyset'$ which dominates every total function $f \leq_T A$.*

Corollary 7.15 (Nonlow$_2$ Escape Theorem)

Fix $A \leq_T \emptyset'$. Then A is nonlow$_2$ *($A'' \not\leq_T \emptyset''$) iff for every function $g \leq_T \emptyset'$ there is a function $f \leq_T A$ which escapes g, i.e.,*

$$(\forall g \leq_T \emptyset')(\exists f \leq_T A)(\exists^\infty x)[g(x) \leq f(x)].$$

computably dominated sets

Definition 7.16

A is called *computably dominated* if each function $g \leq_{\mathrm{T}} A$ is dominated by a computable function.

Definition 7.17

$E \subseteq \mathbb{N}$ is *hyperimmune* if E is infinite and its principal function p_E is not dominated by a computable function.

Theorem 7.18

A is not computably dominated iff there is a hyperimmune set $E \equiv_{\mathrm{T}} A$.

Proof.

"$\Leftarrow$": Immediate since $p_E \leq_{\mathrm{T}} E$.
"$\Rightarrow$": Suppose $g \leq_{\mathrm{T}} A$ is not dominated by a computable function. Let $E = \mathrm{ran}(h)$, where the function h is defined as follows: $h(0) = 0$, and for each $n \in \mathbb{N}$, $h(2n+1) = h(2n) + g(n) + 1$ and $h(2n+2) = h(2n+1) + p_A(n) + 1$. Clearly $E \equiv_{\mathrm{T}} h \equiv_{\mathrm{T}} A$. Moreover $g(n) \leq h(2n+1)$, so that $h = p_E$ is not dominated by a computable function. □

A computably dominated set is usually called a set of *hyperimmune-free degree*.

computably dominated sets

Proposition 7.19

$f \leq_{tt} A$ iff there is a Turing functional Ψ and a computable function t such that $f = \Psi^A$ and the number of steps needed to compute $\Psi^A(n)$ is bounded by $t(n)$.

Proof.

"$\Rightarrow$": Suppose $f = \Phi^A$ and Φ^Z is total for each oracle Z. Let $t(n)$ be the maximal among the running times of the computation $\Phi^Z(n)$ for each oracle Z. Note that only a finite number of strings σ can be involved in the oracles by the use principal.
"$\Leftarrow$": Let $\widetilde{\Psi}$ be the Turing functional such that $\widetilde{\Psi}^Z(n) = \Psi^Z_{t(n)}(n)$ if the latter is defined, and $\widetilde{\Psi}^Z(n) = 0$ otherwise. Then $\widetilde{\Psi}^Z$ is total for each Z and $\widetilde{\Psi}^A = \Psi^A$. Hence $f \leq_{tt} A$. □

Proposition 7.20

A is computably dominated iff for each function f, $f \leq_T A \;\Rightarrow\; f \leq_{tt} A$.

Proof.

"$\Rightarrow$": Suppose $f = \Psi^A$. Let $g(x) = \mu s[\Psi^A_s(x)\downarrow]$. Then $g \leq_T A$, so there is a computable function t such that $t(x) \geq g(x)$ for each x. Thus $f \leq_{tt} A$ by Proposition 7.19.
"$\Leftarrow$": By Proposition 7.19, each function $f \leq_{tt} A$ is dominated by a computable function. □