Foundations of Computability Theory (8)

Nan Fang

Institute of Software, Chinese Academy of Sciences

Peking University

2025 Autumn Semester

summary of domination and escape properties

Theorem 8.1

For every degree $\mathbf{a} \leq_T \mathbf{0}'$,

1. $\mathbf{a} = \mathbf{0}' \quad \Leftrightarrow \quad (\exists g \leq_T \mathbf{a})[g \text{ dominates all p.c. functions}]$;
2. $\mathbf{a} <_T \mathbf{0}' \quad \Leftrightarrow \quad (\forall g \leq_T \mathbf{a})(\exists \theta \text{ p.c.})[\theta \text{ escapes } g]$;
3. $\mathbf{a} \in \mathbf{L}_2 \quad (i.e., \mathbf{a}'' \leq_T \mathbf{0}'') \quad \Leftrightarrow \quad (\exists g \leq_T \mathbf{0}')(\forall f \leq_T \mathbf{a})[g \text{ dominates } f]$;
4. $\mathbf{a} \in \overline{\mathbf{L}}_2 \quad (i.e., \mathbf{a}'' \not\leq_T \mathbf{0}'') \quad \Leftrightarrow \quad (\forall g \leq_T \mathbf{0}')(\exists f \leq_T \mathbf{a})[f \text{ escapes } g]$.

For every degree $\mathbf{b}$,

5. $\mathbf{b} \in \mathbf{H}_1 \quad (i.e., \mathbf{0}'' \leq_T \mathbf{b}') \quad \Leftrightarrow \quad (\exists g \leq_T \mathbf{b})(\forall f \leq_T \mathbf{0})[g \text{ dominates } f]$;
6. $\mathbf{b} \in \overline{\mathbf{H}}_1 \quad (i.e., \mathbf{0}'' \not\leq_T \mathbf{b}') \quad \Leftrightarrow \quad (\forall g \leq_T \mathbf{b})(\exists f \leq_T \mathbf{0})[f \text{ escapes } g]$;
7. $\mathbf{b} \in \overline{\mathbf{HIF}} \quad \Leftrightarrow \quad (\exists g \leq_T \mathbf{b})(\forall f \leq_T \mathbf{0})[g \text{ escapes } f]$.
8. $\mathbf{b} \in \mathbf{HIF} \quad \Leftrightarrow \quad (\forall g \leq_T \mathbf{b})(\exists f \leq_T \mathbf{0})[f \text{ dominates } g]$;

Proposition 8.2

If A is in Δ^0_2 and incomputable, then A is not computably dominated.

Proof.

Let $\{A_s\}_{s\in\omega}$ be a computable approximation of A. Then the following function g is total:

$$g(s) = (\mu t \geq s)\ [A \upharpoonright s = A_t \upharpoonright s].$$

Note that $g \leq_{\mathrm{T}} A$. Assume that A is computably dominated, then there is a computable function f that *majorizes* g. We show that in this case A is computable:

For each n and each $s > n$ we have $A_t(n) = A(n)$ for $t = g(s) \in [s, f(s))$. On the other hand, if s is sufficiently large then $A_u(n) = A_s(n)$ for all $u \geq s$. Thus, to compute A, on input n we find the least $s > n$ such that $A_u(n) = A_s(n)$ for all $u \in [s, f(s))$. Then $A(n) = A_{g(s)}(n) = A_s(n)$. □

binary trees

Definition 8.3

1. A *binary tree* $T \subseteq s^{<\omega}$ is a set of binary strings closed under initial segments, i.e., $\sigma \in T$ and $\tau \prec \sigma$ imply $\tau \in T$. Fix any tree T.
2. A *path* through the tree T is an infinite binary sequence $P \in 2^{\omega}$ such that all its finite initial segments are in T. The class of all paths through T is denoted by $[T]$.
3. A node $\sigma \in T$ is *extendible* if the set $\{\tau \in T \colon \sigma \preceq \tau\}$ is infinite. The *subtree of extendible nodes* of T is denoted by T^{ext}.

Weak König's lemma

Lemma 8.4 (Weak König's lemma)

Let $T \subseteq 2^{<\omega}$ be an infinite binary tree, that is, such that $|T| = \infty$. Then, $[T]$ is non-empty.

Proof.

We construct a path X by induction on n. As T is infinite, by the pigeonhole principle there exists $i \in \{0, 1\}$ and an infinity of nodes $\sigma \in T$ which extend i (i.e. with $i \prec \sigma$). We define $X(0) = i$. Suppose that $\tau = X(0)X(1) \dots X(n)$ is defined with $\tau \in T$ and such that there is an infinity of nodes $\sigma \in T$ for which $\tau \subseteq \sigma$. By the pigeonhole principle, there exists $i \in \{0, 1\}$ and an infinity of nodes $\sigma \in T$ which extend $\tau^\frown i$. We define $X(n+1) = i$.

By induction on n, we thus define a set X such that $X \upharpoonright n \in T$ for all n. □

Cantor space 2^ω

Given a string $\sigma \in 2^{<\omega}$, we write $[\![\sigma]\!]$ for the class $\{X \in 2^\omega : X \prec \sigma\}$. We will call *cylinder* a class of the form $[\![\sigma]\!]$.

Definition 8.5

The *open* classes of Cantor space $2^{\mathbb{N}}$ are arbitrary unions of cylinders, i.e., sets of the form $[\![W]\!] := \bigcup_{\sigma \in W} [\![\sigma]\!]$ for a set $W \in 2^{<\omega}$. The *closed* classes are the complements of the open classes.

Proposition 8.6

A finite intersection of open classes is open. By passing to the complement a finite union of closed classes is closed.

Proposition 8.7

A class $\mathcal{P} \in 2^\omega$ is closed iff there is a tree $T \subseteq 2^{<\omega}$ such that $P = [T]$.

Compactness

Proposition 8.8

Let $\mathcal{P}_0 \supseteq \mathcal{P}_1 \supseteq \ldots$ be a decreasing sequence of non-empty closed classes. Then, $\bigcap_n \mathcal{P}_n$ is non-empty.

Proof.

Let T_n be a tree such that $[T_n] = \mathcal{P}_n$. We can assume without loss of generality $T_{n+1} \subseteq T_n$. We have $\bigcap_n [T_n] = [\bigcap_n T_n]$ by the following observation.

$$X \in \bigcap_n [T_n] \quad \Leftrightarrow \quad (\forall n)(\forall m)\ [X \upharpoonright m \in T_n] \quad \Leftrightarrow \quad (\forall m)(\forall n)\ [X \upharpoonright m \in T_n] \quad \Leftrightarrow \quad X \in [\bigcap_n T_n].$$

Let $T = \bigcap_n T_n$. In particular $[T] = \bigcap_n \mathcal{P}_n$.

Suppose absurdly that $[T] = \bigcap_n \mathcal{P}_n$ is empty. According to König's lemma there is therefore an integer a such that no string σ of size greater than or equal to a is in T. As the strings of size a are in finite quantity and the sequence $(T_n)_{n \in \omega}$ is decreasing by inclusion, there must therefore be integer m such that none of these strings belong to T_m. So $[T_m]$ is empty, which contradicts the assumptions. □

Σ^0_1 and Π^0_1 classes

Definition 8.9

A class $\mathcal{U} \subseteq 2^\omega$ is called Σ^0_1 if there exists a c.e. set $W \subseteq 2^{<\omega}$ such that $\mathcal{U} = \bigcup_{\sigma \in W} [\![\sigma]\!]$. A class $\mathcal{P} \subseteq 2^\omega$ is said to be Π^0_1 if its complement is a Σ^0_1 class.

Proposition 8.10

Σ^0_1 *classes are closed under finite intersection. By passing to the complement, the* Π^0_1 *classes are closed under finite union.*

Proof.

Let $\mathcal{U}_0 = \bigcup_{\sigma \in W_0} [\![\sigma]\!]$ and $\mathcal{U}_1 = \bigcup_{\sigma \in W_1} [\![\sigma]\!]$ be two Σ^0_1 classes. Then, the open class $\mathcal{U}_0 \cap \mathcal{U}_1$ is described by the c.e. set which lists the longest string among σ_0, σ_1 for any $\sigma_0 \in W_0$ and $\sigma_1 \in W_1$ such that $\sigma_0 \subseteq \sigma_1$ or such that $\sigma_1 \subseteq \sigma_0$. The string thus enumerated corresponds to $[\![\sigma_0]\!] \cap [\![\sigma_1]\!]$. □

characterization of Π^0_1 classes

Proposition 8.11

A class $\mathcal{P}$ is Π^0_1 iff there is a computable tree $T \subseteq 2^{<\omega}$ such that $[T] = \mathcal{P}$.

Proof.

"$\Leftarrow$": Let $T \subseteq 2^{<\omega}$ be a computable tree. The class $[T] = \{X : \forall n\, X \upharpoonright n \in T\}$ is Π^0_1. Indeed its complement is the Σ^0_1 class $\bigcup_{\sigma \in \overline{T}} [\![\sigma]\!]$

"$\Rightarrow$": Suppose that $\mathcal{P}$ is Π^0_1. Let $\mathcal{U} = \bigcup_{\sigma \in U} [\![\sigma]\!]$ be its complement. We compute the following tree $T \subseteq 2^{<\omega}$:
At stage t, for any string $\sigma \in 2^{<\omega}$ of size t, we decide $\sigma \in T$ iff for any prefix $\tau \preceq \sigma$ we have $\tau \notin U_t$.

It is clear that T is closed under prefix: if σ of size t is in T then no prefix of σ is in U at stage t, so for $s \leq t$, also no prefix of $\sigma \upharpoonright s$ is in U at stage s.
Now if $X \in \mathcal{P}$ then no prefix of X is in U and therefore each of those prefixes will be in T.
Conversely if $X \notin \mathcal{P}$ then a prefix σ of X goes into U at a certain stage t. By construction no strings $\tau \succeq \sigma$ longer than t will be in T. So $\mathcal{P} = [T]$. □

characterization of Σ^0_1 and Π^0_1 classes

Proposition 8.12

Let $\mathcal{P} \subseteq 2^\omega$ be a class.

(i) *$\mathcal{P}$ is Σ^0_1 iff $\mathcal{P} = \{X \in 2^\omega : \exists n\, R(X \upharpoonright n)\}$ for a computable predicate $R \subseteq 2^{<\omega}$.*

(ii) *$\mathcal{P}$ is Π^0_1 iff $\mathcal{P} = \{X \in 2^\omega : \forall n\, R(X \upharpoonright n)\}$ for a computable predicate $R \subseteq 2^{<\omega}$.*

Proof.

For (ii), "$\Rightarrow$": given a Π^0_1 class $\mathcal{P}$, it suffices to consider the computable tree T such that $[T] = \mathcal{P}$. The computable predicate is simply T.

"$\Leftarrow$": If $\mathcal{P} = \{X \in 2^\omega : \forall n\, R(X \upharpoonright n)\}$ then the computable tree $T = \{\sigma \colon \forall \tau \preceq \sigma\ R(\tau)\}$ is such that $[T] = \mathcal{P}$.

We obtain (i) by passing to the complement. □

existence of Π^0_1 classes with no computable members

Proposition 8.13

There exists a nonempty Π^0_1 class with no computable members.

Proof.

We define the class

$$\mathcal{P} = \{X \in 2^\omega : \forall e \, \forall t \ (\Phi_{e,t}(e) \uparrow \vee \Phi_{e,t}(e) \downarrow \neq X(e))\}.$$

The class $\mathcal{P}$ contains all the sets X such that $X(n)$ can take any value if $\Phi_n(n) \uparrow$, and which are always different from $\Phi_n(n)$ if $\Phi_n(n) \downarrow$. It is clear that this class is not empty (the halting problem for example easily computes an element of $\mathcal{P}$). By Proposition 8.12, $\mathcal{P}$ is a Π^0_1 class. Moreover, the class $\mathcal{P}$ does not contain any computable set: if X is computable then there must be some e such that $\Phi_e(e) \downarrow = X(e)$. $\square$

Low Basis Theorem

Note that $\emptyset'$ can decide whether a computable tree $G \subseteq 2^{<\omega}$ is finite because

$$|G| < \infty \quad \Leftrightarrow \quad (\exists n)(\forall \sigma \in 2^n)\ [\sigma \notin G].$$

Theorem 8.14 (Low Basis Theorem, Jockusch & Soare 1972)

Every nonempty Π^0_1 class P contains a low set Y (i.e., $Y' \leq_T \emptyset'$).

Proof.

Let T be a computable tree such that $[T] = C$. Use $\emptyset'$ to define a sequence of infinite computable trees $T = T_0 \supseteq T_1 \supseteq \dots$ as follows. Define:

$$U_e = \left\{\sigma : \Phi^{\sigma}_{e,|\sigma|}(\mathrm{e}) \uparrow\right\}, \tag{1}$$

which is also a computable tree. Given T_e, use $\emptyset'$ to check whether $T_e \cap U_e$ is infinite.
(a) If it is infinite, let $Z(n) = 1$ and define $T_{e+1} = T_e \cap U_e$. Note that in this case $\Phi^X_e(e) \uparrow$ for all $X \in [T_{e+1}]$.
(b) If it is finite, let $Z(n) = 0$ and define $T_{e+1} = T_e$. Note that in this case $\Phi^X_e(e) \downarrow$ for all $X \in [T_{e+1}]$.

By compactness, $\bigcap_{e\in\omega}[T_e]$ is nonempty. Choose $Y \in \bigcap_{e\in\omega}[T_e]$. Then by construction, $Y' = Z \leq_T \emptyset'$. Therefore, Y is low. □

Comnputably Dominated Basis Theorem

Theorem 8.15 (Comnputably Dominated Basis Theorem)

Every nonempty Π^0_1 class $\mathcal{P}$ contains a computably dominated set Y.

Proof.

We define a descending sequence of nonempty Π^0_1 classes $(\mathcal{P}_e)_{e\in\omega}$ and use compactness to conclude that there is $Y \in \bigcap_e \mathcal{P}_e$.

Let $\mathcal{P}_0 = \mathcal{P}$.
At stage $e+1$. Suppose $\mathcal{P}_e$ has been determined. Let $\mathcal{Q}_x = \mathcal{P}_e \cap \{Z : \Phi^Z_e(x)\uparrow\}$. Then $\mathcal{Q}_x$ is a Π^0_1 class uniformly in x. Check if there is x such that $\mathcal{Q}_x \neq \emptyset$.
(a) If yes, let x be the least such and let $\mathcal{P}_{e+1} = \mathcal{Q}_x$. Then Φ^Z_e is partial for each $Z \in \mathcal{P}_{e+1}$.
(b) Otherwise let $\mathcal{P}_{e+1} = \mathcal{P}_e$. (We will show that there is a computable function f dominating Φ^Z_e for each $Z \in \mathcal{P}_{e+1}$.)

Let $Y \in \bigcap_e \mathcal{P}_e$. If Φ^Y_e is a total function then at stage $e+1$, we are in case (b) since $Y \in \mathcal{P}_{e+1}$. To compute a function f that dominates Φ^Y_e, let T be the computable tree such that $\mathcal{P}_e = [T]$. Given x, let $f(x)$ be the least s such that $\Phi^\sigma_{e,s}(x)\downarrow$ for all $\sigma \in T \cap 2^s$. Such an s exists, otherwise the class $\mathcal{Q}_x$ considered at stage $e+1$ is nonempty. Note that $f(x)$ is greater than all the values $\Phi^\sigma_{e,s}(x)\downarrow$ for all $\sigma \in T\cap 2^s$. Then f dominates Φ^Y_e. $\square$

Corollaries of Basis Theorem

Corollary 8.16

There is a low set which is not computable.

Corollary 8.17

There is a computably dominated set which is not computable.

Cone Avoidance Basis Theorem

Theorem 8.18

Let X be a non-computable set and $\mathcal{P}$ a non-empty Π^0_1 class. Then, there exists an element of $\mathcal{P}$ which does not compute X.

Proof.

- *Case 1:* X is Δ^0_2. By Proposition 8.2, X is not computably dominated. By the computably dominated basis theorem, $\mathcal{P}$ contains a computably dominated set P. In particular, P does not compute X.
- *Case 2:* X is not Δ^0_2. By the low basis theorem, $\mathcal{P}$ contains a low set, so $P \in \Delta^0_2$. In particular, P does not compute X.

In each case, $\mathcal{P}$ contains an element which does not compute X. □