Foundations of Computability Theory (9)

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Proposition 9.1

Let $Q_{e,x} = \{\sigma : \Phi^{\sigma}_{e,|\sigma|}(x)\uparrow\}$ *and* $\mathcal{Q}_{e,x} = \{Y : \Phi^{Y}_{e}(x)\uparrow\}$. *Then* $Q_{e,x}$ *is a computable tree,* $\mathcal{Q}_{e,x}$ *is a* Π^0_1 *class, and* $\mathcal{Q}_{e,x} = [Q_{e,x}]$.

Proof.

Clearly, $Q_{e,x}$ is computable.
For $\sigma \preceq \tau$, by use principle, if $\Phi^{\sigma}_{e,|\sigma|}(x)\downarrow$ it must be that $\Phi^{\tau}_{e,|\tau|}(x)\downarrow$. Thus, if $\tau \in Q_{e,x}$, then $\sigma \in Q_{e,x}$. Hence, $Q_{e,x}$ is a tree.

$$
\begin{aligned}
Y \notin \mathcal{Q}_{e,x} \quad &\Leftrightarrow \quad \Phi^{Y}_{e}(x)\downarrow \\
&\Leftrightarrow \quad (\exists s)(\exists n)[\Phi^{Y\restriction n}_{e,s}(x)\downarrow] \\
&\Leftrightarrow \quad (\exists t)[\Phi^{Y\restriction t}_{e,t}(x)\downarrow)] \\
&\Leftrightarrow \quad (\exists t)[Y\restriction t \notin Q_{e,x}] \\
&\Leftrightarrow \quad Y \notin [Q_{e,x}].
\end{aligned}
$$

$\square$

Low Basis Theorem

$\emptyset'$ can decide whether a computable tree $G \subseteq 2^{<\omega}$ is finite (or a Π^0_1 class is empty) because

$$|G| < \infty \quad \Leftrightarrow \quad (\exists n)(\forall \sigma \in 2^n)\ [\sigma \notin G].$$

Theorem 9.2 (Low Basis Theorem, Jockusch & Soare 1972)

Every nonempty Π^0_1 class $\mathcal{P}$ contains a low set Y (i.e., $Y' \leq_T \emptyset'$).

Proof.

We define a descending sequence of nonempty Π^0_1 classes $(\mathcal{P}_e)_{e\in\omega}$ as follows.

Let $\mathcal{P}_0 = \mathcal{P}$.
At stage $e+1$. Suppose $\mathcal{P}_e$ has been determined. Use $\emptyset'$ to check if $\mathcal{P}_e \cap \mathcal{Q}_{e,e} \neq \emptyset$.
(a) If yes, let $Z(e) = 0$ and let $\mathcal{P}_{e+1} = \mathcal{P}_e \cap \mathcal{Q}_{e,e}$. Note that in this case for all $X \in \mathcal{P}_{e+1}$, $X \in \mathcal{Q}_{e,e}$, i.e., $\Phi^X_e(e)\uparrow$.
(b) Otherwise let $Z(e) = 1$ and let $\mathcal{P}_{e+1} = \mathcal{P}_e$. Note that in this case for all $X \in \mathcal{P}_{e+1}$, $X \notin \mathcal{Q}_{e,e}$, i.e., $\Phi^X_e(e)\downarrow$.

By compactness there exists $Y \in \bigcap_e \mathcal{P}_e$. By construction, $Y' = Z \leq_T \emptyset'$. Therefore, Y is low. $\square$

Comnputably Dominated Basis Theorem

Theorem 9.3 (Comnputably Dominated Basis Theorem)

Every nonempty Π^0_1 class $\mathcal{P}$ contains a computably dominated set Y.

Proof.

We define a descending sequence of nonempty Π^0_1 classes $(\mathcal{P}_e)_{e \in \omega}$ as follows.

Let $\mathcal{P}_0 = \mathcal{P}$.
At stage $e+1$. Suppose $\mathcal{P}_e$ has been determined. Check if there is x such that $\mathcal{P}_e \cap \mathcal{Q}_{e,x} \neq \emptyset$.
(a) If yes, let x be the least such and let $\mathcal{P}_{e+1} = \mathcal{P}_e \cap \mathcal{Q}_{e,x}$. Then Φ_e^X is partial for each $X \in \mathcal{P}_{e+1}$.
(b) Otherwise let $\mathcal{P}_{e+1} = \mathcal{P}_e$. (We will show that there is a computable function f dominating Φ_e^X for each $X \in \mathcal{P}_{e+1}$.)

By compactness there exists $Y \in \bigcap_e \mathcal{P}_e$. If Φ_e^Y is a total function then at stage $e+1$, we are in case (b) since $Y \in \mathcal{P}_{e+1}$, i.e., $\mathcal{P}_e \cap \mathcal{Q}_{e,x} = \emptyset$ for all x. Let T be the computable tree such that $\mathcal{P}_e = [T]$. Fix x, then $T \cap Q_{e,x}$ is a finite tree. Thus, there is an s such that $\sigma \notin T \cap Q_{e,x}$ for all $\sigma \in 2^s$, i.e., $\Phi_{e,s}^{\sigma}(x)\downarrow$ for all $\sigma \in T \cap 2^s$. Let $f(x)$ be the least such s, then $f(x)$ is greater than all the values $\Phi_{e,s}^{\sigma}(x)\downarrow$ for $\sigma \in T \cap 2^s$. Thus, f dominates Φ_e^Y.

$\square$

Corollaries of Basis Theorem

Corollary 9.4

There is a low set which is not computable.

Corollary 9.5

There is a computably dominated set which is not computable.

Cone Avoidance Basis Theorem

Theorem 9.6

Let X be a non-computable set and $\mathcal{P}$ a non-empty Π^0_1 class. Then, there exists an element of $\mathcal{P}$ which does not compute X.

Proof.

- *Case 1:* X is Δ^0_2. By Proposition 8.2, X is not computably dominated. By the computably dominated basis theorem, $\mathcal{P}$ contains a computably dominated set P. In particular, P does not compute X.
- *Case 2:* X is not Δ^0_2. By the low basis theorem, $\mathcal{P}$ contains a low set, so $P \in \Delta^0_2$. In particular, P does not compute X.

In each case, $\mathcal{P}$ contains an element which does not compute X. □

Post's Problem

Post's Problem (1944)

Whether there exists a c.e. set A with $\emptyset <_{\mathrm{T}} A <_{\mathrm{T}} \emptyset'$?

simple sets

Definition 9.7

1. An infinite set is *immune* if it contains no infinite c.e. set.
2. A c.e. set A is *simple* if $\overline{A}$ is immune.

Proposition 9.8

If A is simple then:

1. *A is not computable;*
2. *A is not creative;*
3. *A is not m-complete (i.e., $K \not\leq_{\mathrm{m}} A$).*

Proof.

1. $\overline{A}$ cannot be c.e. since otherwise it would contain an infinite c.e. set.
2. The complement of a creative set does contain an infinite c.e. set by Theorem 3.10.
3. If $K \leq_{\mathrm{m}} A$, then A is creative by Corollary 3.14.

$\square$

Post's simple set

Theorem 9.9 (Post's simple set, 1944)

There exists a simple set A.

Proof.

We construct a coinfinite c.e. set $A = \bigcup_s A_s$ by stages such that for all e we meet the following requirement:

$$P_e : \ |W_e| = \infty \quad \Rightarrow \quad W_e \cap A \neq \emptyset. \tag{1}$$

Construction. Let $A_0 = \emptyset$. At stage $s+1$. Given A_s, choose the least $e < s$ such that

$$W_{e,s} \cap A_s = \emptyset \quad \& \quad (\exists x > 2e)[x \in W_{e,s}]. \tag{2}$$

If e exists, choose the least corresponding x and enumerate x in A. We say P_e receives attention at stage $s+1$. If there is no such e, go to stage $s+2$.

Verification. *Claim 1:* $\overline{A}$ is infinite.
Note that each P_i receives attention at most once thus enumerating at most one element into A greater than $2i$. Hence, there are at most e elements in $A \restriction (2e+1)$, $|\overline{A} \restriction (2e+1)| \geq e+1$.

Claim 2: Every requirement P_e is satisfied.
We prove by induction on e. For $e \geq 0$, by induction choose s_0 such that no $P_i, i < e$, satisfies (2) at any stage $s > s_0$. If W_e is infinite, choose $x > 2e$ satisfying (2) at some stage $s > s_0$. Hence, P_e receives attention at stage s, and $A \cap W_e \neq \emptyset$ thereafter. $\square$

Definition 9.10

1. Given a finite set $A = \{x_1, x_2, \ldots, x_k\}$ we define $y = 2^{x_1} + 2^{x_2} + \cdots + 2^{x_k}$ to be the *canonical index* of A. Let D_y denote the finite set with canonical index y and D_0 denote $\emptyset$.

2. A sequence $\{F_n\}_{n\in\omega}$ of finite sets is a *strong array* if there is a computable function f such that $F_n = D_{f(n)}$.

3. An array is *disjoint* if its members are pairwise disjoint.

hypersimple sets

Theorem 9.11 (Kuznecov, Medvedev, Uspenskii)

An infinite set A is hyperimmune iff for any disjoint strong array $\{F_n\}_{n\in\omega}$ there exists n such that $F_n \cap A = \emptyset$.

Proof.

"$\Rightarrow$": We prove the contrapositive. Suppose $\{D_{g(x)}\}_{x\in\omega}$ is a disjoint strong array with $D_{g(x)} \cap A \neq \emptyset$ for all x. Define $f(x) = 1 + \max(\bigcup\{D_{g(y)}\}_{y\leq x})$. Then $f(x) > p_A(x)$ for all x. Therefore, A is not hyperimmune.

"$\Leftarrow$": We prove the contrapositive. Assume p_A is majorizes by a computable function f. We define by induction a disjoint strong array $\{D_{g(n)}\}_{n\in\omega}$ such that $D_{g(n)} \cap A \neq \emptyset$ for all n. Let $D_{g(0)} = [0, f(0)]$. Given $g(0), g(1), \dots, g(n)$, define $k = 1 + \max(\bigcup\{D_{g(i)}\}_{i\leq n})$ and define $D_{g(n+1)} = [k, f(k)]$. Now $k \leq p_A(k) \leq f(k)$ so $p_A(k) \in D_{g(n+1)} \cap A$. □

Definition 9.12

A c.e. set A is *hypersimple*, if $\overline{A}$ is hyperimmune.

$$\begin{aligned} A \text{ is hyperimmune} &\quad\Rightarrow\quad A \text{ is immune} \\ A \text{ is hypersimple} &\quad\Rightarrow\quad A \text{ is simple} \end{aligned}$$

Dekker's Theorem

Definition 9.13

Let A be an infinite c.e. set and f a $1:1$ computable function with range A (an *enumeration* of A). Let a_s denote $f(s)$. Define the deficiency set D to be the Σ_1 set of deficiency stages:

$$D = \{s : (\exists t > s)[a_t < a_s]\}.$$

Theorem 9.14 (Dekker, 1954)

Let A be any noncomputable c.e. set and D the deficiency set of A with respect to some fixed enumeration of A.

1. $A \leq_{\mathrm{T}} D$.
2. $D \leq_{\mathrm{tt}} A$.
3. *D is hypersimple.*

Corollary 9.15

For every noncomputable c.e. set A there is an hypersimple set $D \equiv_{\mathrm{T}} A$. Hence, every nonzero c.e. degree contains an hypersimple set.

Proof of Dekker's Theorem

1. First we show that $\overline{D}$ is infinite. Suppose otherwise, let $s_0 > \max \overline{D}$. Then $t \in D$ for all $t \geq s_0$. In particular, there exists $s_1 > s_0$ such that $a_{s_1} < a_{s_0}$. Similarly, there exists $s_2 > s_1$ such that $a_{s_2} < a_{s_1}$. Continuing in this way we obtain an infinite decreasing sequence which contradicting to the fact that $a_s \geq 0$ for all s.

   Now for each n, let $b_n = a_{P_{\overline{D}}(n)}$. Then $b_0 < b_1 < b_2 < \dots$ is an increasing sequence within A. In particular, $b_n \geq n$ for all n.

   Fix x, let $m = P_{\overline{D}}(x)$. Then $b_x \geq x$ and for all $t > m$, $a_t > a_m = b_x \geq x$. Thus,

   $$x \in A \quad \text{iff} \quad x \in \{a_0, a_1, \dots, a_{P_{\overline{D}}(x)}\} \tag{3}$$

   Hence, $A \leq_{\mathrm{T}} D$.

2. We define a pair (g, h) of computable functions as follows: Let $g(s) = \{0, 1, \cdots, a_s - 1\}$. Let $A_s = \{a_v : v \leq s\}$. For each s, find the least stage v such that $A_v \upharpoonright g(s) = A \upharpoonright g(s)$. Now let $h(s, A \upharpoonright g(s)) = 1$ if $v > s$ and 0 otherwise. Then $D(x) = h(x, A \upharpoonright g(x))$ for all x.

3. D is Σ_1 and hence c.e. If some computable function g dominates $P_{\overline{D}}$, then by (3) $x \in A$ iff $x \in \{a_0, a_1, \dots, a_{g(x)}\}$, which implies that A is computable. Thus, $\overline{D}$ is hyperimmune, and D is hypersimple