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Extending CL-reducibility on array noncomputable degrees

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ABSTRACT

Given a function f, f-bounded-Turing (f-bT-) reducibility is the Turing reducibility with use function bounded by f. In the special case where $f = \mathrm{id} + c$ (with id being the identity function and c a constant), this is referred to as cl-reducibility. In a work by Barmpalias, Fang, and Lewis-Pye, it was proven that there exist two left-c.e. reals such that no left-c.e. real $(\mathrm{id}+g)$-bT-computes both of them whenever g is computable, nondecreasing, and satisfies $\sum_n 2^{-g(n)} = \infty$. Moreover, such maximal pairs exist precisely within every array noncomputable degree. This result generalizes a prior result on cl-reducibility, which states that there exist two left-c.e. reals such that no left-c.e. real cl-computes both of them. An open question remained as to whether a similar extension could apply to another result on cl-reducibility, which asserts that there exists a left-c.e. real not cl-reducible to any random left-c.e. real. We answer this question affirmatively, providing a simpler proof compared to previous works. Additionally, we streamline the proof of the initial extension.

1. Introduction

In the field of algorithmic randomness, cl-reducibility plays an important role. This reducibility is a variant of Turing reducibility where the use function is bounded from above by $\mathrm{id} + c$ where id is the identity function and c is some constant. Accordingly, in case some real α is cl-reducible to a real β, we also say that β cl-computes α. Basic results about cl-reducibility are the two following theorems due to Yu and Ding [17], and to Barmpalias and Lewis [1]. Note that throughout this paper, "random" always refers to Martin-Löf randomness.

Theorem 1.1 *(Yu and Ding [17]). There are two left-c.e. reals such that no left-c.e. real cl-computes both of them.*

Theorem 1.2 *(Barmpalias and Lewis [1]). There is a left-c.e. real such that no left-c.e. random real cl-computes it.*

Barmpalias et al. [6] showed that the left-c.e. reals claimed in Theorems 1.1 and 1.2 exist precisely in the array noncomputable c.e. Turing degrees, hence both theorems yield characterizations of these degrees.

Theorem 1.3 *(Barmpalias et al. [6]). The following are equivalent for a c.e. Turing degree* **d**.

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i) There are two left-c.e. reals in **d** *such that no left-c.e. real cl-computes both of them.*
ii) There is a left-c.e. real in **d** *such that no left-c.e. random real cl-computes it.*
iii) **d** *is array noncomputable.*

Given a function f, the restriction of Turing reducibility where the use function is bounded from above by f is denoted by f-bounded-Turing (f-bT-) reducibility, and similarly, if some real α is f-bT-reducible to a real β, we also say that β f-bT-computes α. Barmpalias et al. [7] extended Theorem 1.1 from cl-reducibility to $(\mathrm{id}+g)$-bT-reducibility for some computable function g that is slow-growing. Here a function $g\colon \mathbb{N} \mapsto \mathbb{N}$ is *slow-growing* if it is nondecreasing and $\sum_{n\in\omega} 2^{-g(n)} = \infty$, and is *fast-growing* if it is nondecreasing and $\sum_{n\in\omega} 2^{-g(n)} < \infty$.

Theorem 1.4 *(Barmpalias et al. [7]). Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a computable slow-growing function. There is a pair of left-c.e. reals such that no left-c.e. real $(\mathrm{id}+g)$-bT-computes both of them. Moreover, such a pair exists in every array noncomputable c.e. Turing degree.*

Note that constant functions are computable slow-growing functions, hence cl-reducibility is a special case of $(\mathrm{id}+g)$-bT-reducibility for some computable slow-growing function g. Note that the latter reducibility is not transitive in general.

Barmpalias et al. [7] conjectured that Theorem 1.2 allows a similar extension. We verify their conjecture by the following theorem.

Theorem 1.5. *Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a computable slow-growing function. There is a left-c.e. real such that no left-c.e. random real $(\mathrm{id}+g)$-bT-computes it. Moreover, such a real exists in every array noncomputable c.e. Turing degree.*

With Theorem 1.5, we actually complete the following line of research. First, on the one hand, combined with Theorem 1.4 and Theorem 1.3, we obtain the following corollary.

Corollary 1.6. *Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a computable slow-growing function. The following are equivalent for a c.e. Turing degree* **d**.

i) There are two left-c.e. reals in **d** *such that no left-c.e. real $(\mathrm{id}+g)$-bT-computes both of them.*
ii) There is a left-c.e. real in **d** *such that no left-c.e. random real $(\mathrm{id}+g)$-bT-computes it.*
iii) **d** *is array noncomputable.*

On the other hand, there is no analog of Corollary 1.6 for fast-growing functions, as demonstrated by the following result from Barmpalias et al. [7].

Theorem 1.7 *(Barmpalias et al. [7]). Every left-c.e. real is $(\mathrm{id}+g)$-bT-reducible to any left-c.e. random real for any computable fast-growing function g.*

Thus, we have the following complete dichotomy spectrum. Recall that ibT-reducibility is id-bT-reducibility, which is a special case of cl-reducibility.

- In every array computable c.e. degree, there is no maximal pair of left-c.e. reals with respect to ibT-reducibility and every left-c.e. real is ibT-reducible to some left-c.e. random real. (This comes from the proof of Theorem 1.3 in [6].)
- In every array noncomputable c.e. degree, for every computable nondecreasing function g,
  - if g is fast-growing, then every left-c.e. real is $(\mathrm{id}+g)$-bT-reducible to any left-c.e. random real;
  - if g is slow-growing, then there are two left-c.e. reals that form a maximal pair with respect to $(\mathrm{id}+g)$-bT-reducibility and there is a left-c.e. real not $(\mathrm{id}+g)$-bT-reducible to any left-c.e. random real.

Another line of research related to our results concerns the Kučera-Gács Theorem.

Theorem 1.8 *(Kučera [15], Gács [14]). Every set is wtt-reducible to a random set.*

The use bounds obtained with previous proofs of the Kučera-Gács Theorem [14–16] were largely improved by subsequent studies [3–5,8], and it was demonstrated that the optimal use bounds for which the Kučera-Gács Theorem holds are of the form $\mathrm{id} + g$ where g is a fast-growing function.

Theorem 1.9 *(Barmpalias and Lewis-Pye [3]). Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a computable fast-growing function. Every set is $(\mathrm{id}+g)$-bT-reducible to some random set.*

Theorem 1.10 *(Barmpalias et al. [8]). Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a computable slow-growing function. There exists a set that is not $(\mathrm{id}+g)$-bT-reducible to any random set.*

When considering only left-c.e. reals, Theorem 1.5 can be viewed as the local version of Theorem 1.10 and Theorem 1.7 the local version of Theorem 1.9. Thus, the same optimal use bound holds for the local analog of the Kučera-Gács theorem, while these local theorems do not follow directly from their global counterparts.

The main purpose of this paper is to prove Theorem 1.5. Additionally, we provide a simpler proof of Theorem 1.4 compared to the one in [7]. Our proof of Theorem 1.5 is also simpler than the corresponding part of the proof of Theorem 1.3 in [6]. These simplifications primarily result from a shift in perspective. In constructing left-c.e. reals, we do calculations directly from a global viewpoint, thus avoiding the local and dynamic analysis of two-player games found in previous papers [1,6,7,17] on similar topics.

It is worth mentioning that the results presented here already appeared in the first author's dissertation [13].

In the next section, we begin by recalling some basic notations and establishing necessary conventions. Then, in Section 3, we prove Theorem 1.4 and Theorem 1.5. Both proofs rely on key module lemmas. Finally, in the last section, we discuss some observations and propose open questions for future research.

2. Preliminaries

2.1. Basic notations and conventions

For standard notation and definitions in computability and randomness we refer to the monograph by Downey and Hirschfeldt [11]. In the remainder of this section, we define some notation specific to this article and make some conventions.

The term *string* always refers to a finite binary string. For a string σ, its *length* is denoted by $|\sigma|$ and its positions are numbered by 0 through $|\sigma|-1$, and we write $\sigma=\sigma(0)\cdots\sigma(|\sigma|-1)$. Given two strings σ and τ, we write $\sigma\prec\tau$, in case σ is a proper prefix of τ. Furthermore, σ is lexicographically strictly less than τ, written as $\sigma<_{\mathrm{L}}\tau$, if either $\sigma\prec\tau$ or $\sigma(n)=0$ for the least number n such that $\sigma(n)$ and $\tau(n)$ are both defined and differ. Similar notation such as $>_{\mathrm{L}}$ is used with its natural meaning, in particular, the corresponding nonstrict orderings are denoted by $\preceq$ and $\leq_{\mathrm{L}}$.

In what follows, the term *real* always refers to a real number in the interval $[0,1)$ unless something different is stated explicitly. For our purposes, such a real can be identified with the part of its binary expansion on the right of the decimal point, i.e., with an infinite binary sequence. Such a sequence can in turn be identified with the characteristic sequence of a set of natural numbers. Thus, we use these representations interchangeably.[1]

For a string σ, let $[\![\sigma]\!]$ denote the set of all reals (infinite binary sequences) that extend σ. For a set E of strings, we define $[\![E]\!]=\bigcup_{\sigma\in E}[\![\sigma]\!]$. Let μ denote the Lebesgue measure. For simplicity, we write $\mu([\![E]\!])$ as $\mu(E)$.

For the binary expansion of a real, the positions on the right of the decimal point are numbered by $0,1,2,\ldots$ from left to right. For a real α and a set A of natural numbers, let $\alpha\restriction A$ be the subsequence that corresponds to positions in the set A. That's to say, if $a_0<a_1<a_2<\cdots$ are all the elements of A in magnitude order, then $\alpha\restriction A=\alpha(a_0)\alpha(a_1)\alpha(a_2)\cdots$. We let $\alpha\restriction n=\alpha\restriction\{0,\ldots,n-1\}$, i.e., $\alpha\restriction n$ is the initial segment of α of length n.

Notation *(Truncation).* For a real α and a natural number n, the *truncation* of α to the first n positions, written as $\lfloor\alpha\rfloor_n$, is defined by

$$\lfloor\alpha\rfloor_n=\left\lfloor\alpha\cdot 2^n\right\rfloor\cdot 2^{-n}$$

When viewed as sequence, we always view the truncation of a real α to the first n positions as a finite string of length n, which is exactly $\alpha\restriction n$. This truncation notation was already used by Barmpalias et al. [6]. It is easy to prove the following properties.

Proposition 2.1. *For $\alpha,\beta\in[0,1)$ and $n\in\mathbb{N}$, the following are true.*

1. $\lfloor\alpha+\beta\rfloor_n\geq\lfloor\alpha\rfloor_n+\lfloor\beta\rfloor_n$.
2. *If $\alpha\leq\beta$, then $\lfloor\alpha\rfloor_n\leq\lfloor\beta\rfloor_n$.*
3. *If $\lfloor\alpha\rfloor_n<\lfloor\beta\rfloor_n$, then $\lfloor\alpha\rfloor_n+2^{-n}\leq\lfloor\beta\rfloor_n$.*
4. *$\alpha\restriction n\leq_{\mathrm{L}}\beta\restriction n$ if and only if $\lfloor\alpha\rfloor_n\leq\lfloor\beta\rfloor_n$.*

A *left-c.e. approximation* is a computable nondecreasing sequence of rationals that converges. A real α is *left-c.e.* if there is a left-c.e. approximation $\{\alpha_s\}_{s\in\omega}$ that converges to α. With such an approximation understood, we say α *changed at position n at stage $s+1$* if $\alpha_{s+1}(n)\neq\alpha_s(n)$. Furthermore, note that we may always assume that $\alpha_s=\lfloor\alpha_s\rfloor_s$ without loss of generality.

[1] In the special case of a real that is a dyadic rational, hence has two distinct binary expansions, we always choose the expansion which ends in 0's.

Convention for use functions For the use function of a Turing functional, we make the following convention. When we say a real α is computable from an oracle γ via the Turing functional Φ with use function bounded by h, i.e., α is h-bT-reducible to γ via Φ, it means that the first n bits of α are computable from the first $h(n)$ bits of γ via Φ. In our terminology, it holds that for all n,

$$\alpha \upharpoonright n = \Phi^{\gamma \upharpoonright h(n)} \upharpoonright n, \quad \text{or equivalently,} \quad \lfloor \alpha \rfloor_n = \left\lfloor \Phi^{\lfloor \gamma \rfloor_{h(n)}} \right\rfloor_n.$$

Lemma 2.2. *Let Φ be a Turing functional with use function bounded by h, α and γ be left-c.e. reals with left-c.e. approximations $\{\alpha_i\}_{i\geq 0}$ and $\{\gamma_i\}_{i\geq 0}$. Suppose there are stages $s < t$ such that*

(i) $\alpha_{s-1}(n-1) = \Phi_s^{\gamma_s}(n-1)$ *and* $\alpha_{t-1}(n-1) = \Phi_t^{\gamma_t}(n-1)$*;*
(ii) α *changed at position* $n-1$ *at stage* s *and remained unchanged up to stage* $t-1$.

Then $\gamma_t \geq \lfloor \gamma_s \rfloor_{h(n)} + 2^{-h(n)}$.

Proof. By (ii), $\alpha_{s-1}(n-1) \neq \alpha_{t-1}(n-1)$. Then by (i), we have $\Phi_s^{\gamma_s}(n-1) \neq \Phi_t^{\gamma_t}(n-1)$, which implies $\lfloor \gamma_s \rfloor_{h(n)} \neq \lfloor \gamma_t \rfloor_{h(n)}$. As $\gamma_s \leq \gamma_t$, then $\lfloor \gamma_s \rfloor_{h(n)} < \lfloor \gamma_t \rfloor_{h(n)}$. By Proposition 2.1, we get $\gamma_t \geq \lfloor \gamma_t \rfloor_{h(n)} \geq \lfloor \gamma_s \rfloor_{h(n)} + 2^{-h(n)}$. □

2.2. Array noncomputable c.e. Turing degrees

In the study of c.e. Turing degrees, the concept of array noncomputability was first defined and investigated by Downey et al. [9]. Then it was extended to degrees in general by the same authors in [10]. Array noncomputable degrees are those in which the multiple permitting argument can be effectively applied. Typically, in such an argument, one constructs a finite fragment of the required set for a single requirement, treating it as a module. By appropriately arranging and combining these modules, one can construct a set that satisfies all the requirements. Here we recall the original definition and relevant property of array noncomputability from [9] for later reference.

Definition 2.3 *(Downey et al. [9]).*

(i) A sequence $\mathcal{F} = \{F_n\}_{n\in\omega}$ of finite sets is a *very strong array (v.s.a.)* if the following hold.

   (a) There is a computable function f such that $f(n)$ is the canonical index of F_n;
   (b) $\bigcup_{n\in\omega} F_n = \mathbb{N}$;
   (c) $F_n \cap F_m = \emptyset$ for $n \neq m$;
   (d) $0 < |F_n| < |F_{n+1}|$ for all $n \in \omega$.

(ii) A c.e. set A is *array noncomputable with respect to* the v.s.a. $\mathcal{F}$ (or $\mathcal{F}$-a.n.c. for short) if

$$\forall e\, \exists^\infty n \left(W_e \cap F_n = A \cap F_n\right).$$

(iii) A c.e. degree $\mathbf{a}$ is *array noncomputable* if there is a c.e. set $A \in \mathbf{a}$ which is $\mathcal{F}$-a.n.c. for some v.s.a. $\mathcal{F}$.

Proposition 2.4 *(Downey et al. [9]). Let $\mathbf{d}$ be an array noncomputable c.e. degree and let $\{F_n\}_{n\in\omega}$ be a very strong array. There is a c.e. set $D \in \mathbf{d}$ such that D is $\{F_n\}_{n\in\omega}$-a.n.c.*

3. Main results

Our main proof idea is based on the original idea of the Yu-Ding construction [17], where it was noticed that if γ is a left-c.e. real which could cl-compute both α and β, then alternatively adding little measure to α and β is sufficient to drive γ to be too large. Here we explore this idea further, and generalize it to a framework which will not only simplify the proof of Theorem 1.4 compared to the proof in [7], but could also be adapted to prove Theorem 1.5.

For the analysis of our argument, we describe a process for a left-c.e. real α within interval $[a, b)$.

Definition 3.1. Let $b > a \geq 0$ and $f = 2^{b-a} - 1$. Let α be a left-c.e. real with left-c.e. approximation $\{\alpha_s\}_{s\geq 0}$ such that

(i) $\alpha_0 = 0$;
(ii) there are stages $0 < t_1 < \cdots < t_f$ where 2^{-b} is added to α and for all other stages α remain unchanged.

Then $(0, t_f]$ is called an α $[a, b)$-*loading process* (with loading stages $\{t_i\}_{1\leq i\leq f}$).

Note that at the end of this process, for all positions within $[a,b)$, the bits of α are 1 and for all other positions, the bits are 0. For such a process, it is easy to prove the following lemma.

Lemma 3.2. *If $(0,t_f]$ is an α $[a,b)$-loading process, then for each $n \in [a,b)$, there are 2^{n-a} many loading stages where the least changed position of α is n.*

3.1. Proof of Theorem 1.4

Lemma 3.3. *Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a computable slow-growing function. There is a strictly increasing computable function ℓ such that the following hold. For any triple $\langle \Phi, \Psi, \gamma\rangle$, where Φ, Ψ are Turing functionals with use functions bounded by $\mathrm{id}+g$ and γ is a left-c.e. real in $[0,1)$, uniformly in $\langle \Phi, \Psi, \gamma\rangle$ and in all $a \in \mathbb{N}$, there are left-c.e. reals $\alpha_a, \beta_a \subseteq [a, a+\ell(a)]$ such that*

$$\exists\, x \in [a, a+\ell(a)]\ \big(\alpha_a(x) \neq \Phi^{\gamma}(x) \vee \beta_a(x) \neq \Psi^{\gamma}(x)\big). \tag{1}$$

Proof. Given a computable slow-growing function g, let ℓ be a strictly increasing function such that for all $a \in \mathbb{N}$

$$\sum_{a<n\le a+\ell(a)+1} 2^{-g(n)} \ge 2^{a+1}.$$

As g is slow-growing, such function ℓ exists.

Fix $\langle \Phi, \Psi, \gamma\rangle$ as described and fix $a \in \mathbb{N}$. We uniformly construct two left-c.e. reals $\alpha_a, \beta_a \subseteq [a, a+\ell(a)]$ satisfying (1), which now we refer as requirement $\mathcal{R}$.

Let $h(n) = n + g(n)$, $b = a+\ell(a)+1$ and $q = 2^{b-a}-1$. During the construction, the approximations of $\alpha_a, \beta_a, \Phi, \Psi, \gamma$ at stage s are denoted by $\alpha_{a,s}$, $\beta_{a,s}$, Φ_s, Ψ_s, γ_s, respectively. We say $\mathcal{R}$ *requires attention* at stage s if $s > h(b)$ and (1) fails, i.e.,

$$\alpha_{a,s-1} \restriction [a,b) = \Phi_s^{\gamma_s} \restriction [a,b) \quad \text{and} \quad \beta_{a,s-1} \restriction [a,b) = \Phi_s^{\gamma_s} \restriction [a,b). \tag{2}$$

Our observation in Lemma 3.2 indicates that a conservative strategy of constantly adding 2^{-b} to α could actually cause α to change sometimes at lower positions. This could be helpful to drive γ to grow arbitrary large. However, if we apply any $[a,b)$-loading process on one real α_a trivially, γ could simply copy the behaviors of α_a. But with two reals α_a, β_a at disposal, we can prevent this.

Our strategy works as follows. Whenever $\mathcal{R}$ requires attention we add 2^{-b} to α_a to force γ to change at some position n. If $\mathcal{R}$ requires attention again we do the same for β_a to force γ to change at the same position n. Then after one round, if $\mathcal{R}$ requires attention further, γ would have been forced to change at position n twice, which makes it grow large. In this way, if both of α_a and β_a complete an $[a,b)$-loading process γ would have already been forced to be larger than 1, which contradicts to our assumption that $\gamma < 1$.

Construction: Let $\alpha_{a,0} = \beta_{a,0} = 0$ and $r_0 = 0$. At stage $s > 0$, if $\mathcal{R}$ requires attention and $r_{s-1} < 2q$, we say s is a loading stage and let $r_s = r_{s-1}+1$. Then we check and do the following.

- If r_{s-1} is even, let $\alpha_{a,s} = \alpha_{a,s-1} + 2^{-b}$.
- If r_{s-1} is odd, let $\beta_{a,s} = \beta_{a,s-1} + 2^{-b}$.

Verification:

Claim 3.4. *$\alpha_a, \beta_a \subseteq [a, a+\ell(a)]$, and they are left-c.e. reals, uniformly in a and $\langle \Phi, \Psi, \gamma\rangle$.*

Proof. By construction, clearly α_a, β_a are left-c.e. reals, uniformly in a and $\langle \Phi, \Psi, \gamma\rangle$. Note that there are at most q many loading stages where 2^{-b} is added to α_a while for all other stages α_a stays unchanged. Thus, $\alpha_a \le q\cdot 2^{-b} < 2^{-a}$, which implies $\alpha_a \subseteq [a,b) = [a, a+\ell(a)]$. Similarly, the same holds for β_a as well. □

Claim 3.5. *If $r_s = 2q$ at some stage s, then $\mathcal{R}$ will never require attention at any later stage.*

Proof. For a contradiction, suppose for some s we have $r_s = 2q$ and $\mathcal{R}$ requires attention at some stage $s' > s$. Then there are already $2q$ many loading stages before stage s'. Let $t_1 < t_2 < \cdots < t_{2q}$ be all these loading stages. Let $t_{2q+1} = s'$.

Fix $1 \le i \le q$, by construction stage t_{2i-1} is a loading stage for α_a and stage t_{2i} is a loading stage for β_a. Suppose at stage t_{2i-1} the least changed position of α_a is n, as α_a and β_a add the same measure alternatively, then at stage t_{2i} the least changed position of β_a should also be n. Then by Lemma 2.2 we have

$$\gamma_{t_{2i+1}} \ge \lfloor \gamma_{t_{2i}} \rfloor_{h(n+1)} + 2^{-h(n+1)} \quad \text{and} \quad \gamma_{t_{2i}} \ge \lfloor \gamma_{t_{2i-1}} \rfloor_{h(n+1)} + 2^{-h(n+1)},$$

which implies $\gamma_{t_{2i+1}} > \gamma_{t_{2i-1}} + 2^{-h(n+1)}$.

On the other hand, it is easy to check that $(0, t_{2q-1}]$ is an α_a $[a, b)$-loading process with loading stages $\{t_{2i-1}\}_{1\le i\le q}$. By Lemma 3.2, for each $n \in [a, b)$ there are 2^{n-a} many loading stages where the least changed position of α_a is n. Thus,

$$\gamma_{t_{2q+1}} - \gamma_{t_1} > \sum_{a\le n<b} 2^{n-a} \cdot 2^{-h(n+1)} = 2^{-a-1} \cdot \sum_{a<n\le b} 2^{-g(n)} \ge 1.$$

Then $\gamma \ge \gamma_{t_{2q+1}} > 1$, which contradicts to our assumption that $\gamma < 1$. □

By Claim 3.5, actually $r_{s-1} < 2q$ already holds when $\mathcal{R}$ requires attention at stage s. Thus, by construction there are only finitely many stages where $\mathcal{R}$ requires attention, which means that $\mathcal{R}$ will be satisfied eventually. This completes the proof of Lemma 3.3. □

Now we prove Theorem 1.4. Let $\{\langle \Phi_e, \Psi_e, \gamma_e\rangle\}_{e\in\omega}$ be an effective enumeration of all triples of two Turing functionals with use functions bounded by $\mathrm{id} + g$ and a left-c.e. real in $[0, 1)$. Let ℓ be the function as given by Lemma 3.3. Fix a computable series $\{a_e\}_{e\in\omega}$ of natural numbers such that $a_0 \ge 0$ and $a_{e+1} = a_e + \ell(a_e) + 1$ for all $e \in \omega$. Then uniformly for each $e \in \omega$, by Lemma 3.3 there is a pair of left-c.e. reals $\alpha_e, \beta_e \subseteq [a_e, a_e + \ell(a_e)]$ such that

$$\exists\, x \in [a_e, a_e + \ell(a_e)]\ \big(\alpha_e(x) \ne \Phi_e^{\gamma_e}(x) \vee \beta_e(x) \ne \Psi_e^{\gamma_e}(x)\big). \tag{3}$$

Now let $\alpha = \sum_{e\in\omega} \alpha_e$ and $\beta = \sum_{e\in\omega} \beta_e$. We check that they are as required. First, since α_e, β_e are uniformly left-c.e. in e, α, β are left-c.e. reals. Then fix some $e \ge 0$. Since $\alpha_e, \beta_e \subseteq [a_e, a_e + \ell(a_e)]$ and $a_e + \ell(a_e) < a_{e+1}$, we have

$$\alpha \upharpoonright [a_e, a_e + \ell(a_e)] = \alpha_e \upharpoonright [a_e, a_e + \ell(a_e)] \quad \text{and} \quad \beta \upharpoonright [a_e, a_e + \ell(a_e)] = \beta_e \upharpoonright [a_e, a_e + \ell(a_e)].$$

Thus, with (3) it implies that for all $e \ge 0$,

$$\alpha \ne \Phi_e^{\gamma_e} \quad \text{or} \quad \beta \ne \Psi_e^{\gamma_e}.$$

This proves the first part of the theorem.

For the second part, one can prove by adapting the above construction with standard multiple permitting argument. We will illustrate such an adaption in the proof of Theorem 1.5. For simplicity, we omit the details here.

3.2. *Proof of Theorem 1.5*

Lemma 3.6. *Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a computable slow-growing function. There is a strictly increasing computable function ℓ such that the following hold. For any pair $\langle \Phi, \gamma\rangle$, where Φ is a Turing functional with use function bounded by $\mathrm{id} + g$ and γ is a left-c.e. real in $[0, 1)$, uniformly in $\langle \Phi, \gamma\rangle$ and in all $a \in \mathbb{N}$, there are a left-c.e. real $\alpha_a \subseteq [a, a + \ell(a)]$ and a c.e. set E_a of strings with $\mu(E_a) < 2^{-a}$ such that*

$$\exists\, x \in [a, a + \ell(a)]\ \alpha_a(x) \ne \Phi^{\gamma}(x) \quad \text{or} \quad \gamma \in [\![E_a]\!]. \tag{4}$$

Proof. Given a computable slow-growing function g, let ℓ be a strictly increasing function such that for all $a \in \mathbb{N}$

$$\sum_{a<n\le a+\ell(a)+1} 2^{-g(n)} \ge 2^{2a+2}.$$

As g is slow-growing, such function ℓ exists.

Fix $\langle \Phi, \gamma\rangle$ as described and fix $a \in \mathbb{N}$. We uniformly construct a left-c.e. real $\alpha_a \subseteq [a, a + \ell(a)]$ and a c.e. set E_a with $\mu(E_a) < 2^{-a}$ satisfying (4), which now we refer as requirement $\mathcal{Q}$.

Let $h(n) = n + g(n)$, $b = a + \ell(a) + 1$ and $q = 2^{b-a} - 1$. During the construction, the approximations of $\alpha_a, E_a, \Phi, \gamma$ at stage s are denoted by $\alpha_{a,s}$, $E_{a,s}$, Φ_s, γ_s. We say $\mathcal{Q}$ *requires attention* at stage s if $s > h(b) + a + 1$ and (4) fails, i.e.,

$$\alpha_{a,s-1} \upharpoonright [a, b) = \Phi_s^{\gamma_s} \upharpoonright [a, b) \quad \text{and} \quad \gamma_s \notin [\![E_{a,s-1}]\!]. \tag{5}$$

The same as before, we will use our observation in Lemma 3.2 to drive γ to grow larger than 1. Instead of having two reals α_a, β_a at disposal, now we have a left-c.e. real α_a and a c.e. set E_a. We will apply some partial $[a, b)$-loading process on the real α_a, while prevent γ from copying the behaviors of α_a with the help of E_a.

Our strategy works as follows. Whenever $\mathcal{Q}$ requires attention, before adding 2^{-b} to α_a to force γ to change at some position n, we check the cost of γ (i.e., the least amount γ should increase) to fulfill the change. If the cost is large, we make the change of α_a; otherwise, we force the same change of γ by issuing tests through E_a for the current round and postpone the scheduled change of α_a to the next time whenever $\mathcal{Q}$ requires attention again. In the second case, if $\mathcal{Q}$ requires attention further after the action of α_a, γ would be forced to change at the same position n twice, which makes it grow large. Thus, in both cases the cost of γ are large. By some arrangement, we ensure that during the construction if the measure of E_a exceeds 2^{-a} or α_a complete an $[a,b)$-loading process, γ would be forced to be larger than 1, which contradicts to our assumption that $\gamma < 1$.

For $s > m$, let

$$G(s,m) = \{\sigma \in 2^s : \gamma_s \restriction s \leq_{\mathrm{L}} \sigma <_{\mathrm{L}} (\gamma_s + 2^{-m}) \restriction s\}.$$

Clearly, $\mu(G(s,m)) = 2^{-m}$.

Construction: Let $\alpha_{a,0} = 0$, $E_{a,0} = \emptyset$, and $r_0 = 0$. At stage s, if $\mathcal{Q}$ requires attention and $r_{s-1} < 2q$, supposing by adding 2^{-b} to α_a the least changed position of it would be $n-1$, we check and do the following.

- If r_{s-1} is even and $\gamma_s + 2^{-h(n)-a-1} \geq \lfloor \gamma_s \rfloor_{h(n)} + 2^{-h(n)}$, we say s is a testing stage (for position $n-1$). Then let $E_{a,s} = E_{a,s-1} \cup G(s, h(n)+a+1)$ and $r_s = r_{s-1}+1$.
- Otherwise, we say s is a loading stage (for position $n-1$). Then let $\alpha_{a,s} = \alpha_{a,s-1} + 2^{-b}$ and let r_s be the least even number larger than r_{s-1}.

Verification: By construction, we have the following observations.

- If s is a loading stage then r_s is even and $r_s > 0$.
- If s is a testing stage then r_s is odd and for the next time $\mathcal{Q}$ requires attention that stage must not be a testing stage.

These observations immediately imply the following two claims.

Claim 3.7. *There are at most q many loading stages and at most q many testing stages.*

Claim 3.8. *Between every two testing stages there is at least one loading stage.*

Claim 3.9. *$\alpha_a \subseteq [a, a+\ell(a)]$. E_a is c.e., α_a is left-c.e., uniformly in a and $\langle \Phi, \gamma \rangle$.*

Proof. By construction, clearly E_a is c.e. and α_a is left-c.e., both uniformly in a and $\langle \Phi, \gamma \rangle$. Note that there are at most q many loading stages where 2^{-b} is added to α_a while for all other stages α_a stays unchanged. Thus, $\alpha_a \leq q \cdot 2^{-b} < 2^{-a}$, which implies $\alpha_a \subseteq [a,b) = [a, a+\ell(a)]$. $\square$

Claim 3.10. *If s is a testing stage for position $n-1$, then $\mu(E_{a,s}) - \mu(E_{a,s-1}) \leq 2^{-h(n)-a-1}$. Moreover, if later $\mathcal{Q}$ requires attention at stage t, then $\gamma_t \geq \lfloor \gamma_s \rfloor_{h(n)} + 2^{-h(n)}$.*

Proof. Let s be a testing stage for position $n-1$. Then $E_{a,s} = E_{a,s-1} \cup G(s, h(n)+a+1)$. Thus,

$$\mu(E_{a,s}) - \mu(E_{a,s-1}) \leq \mu(G(s, h(n)+a+1)) = 2^{-h(n)-a-1}.$$

If $\mathcal{Q}$ requires attention at later stage t, then $\gamma_t \notin [\![E_{a,t-1}]\!]$, especially, $\gamma_t \notin [\![G(s,h(n)+a+1)]\!]$. Then $\gamma_t \restriction s \geq_{\mathrm{L}} (\gamma_s + 2^{-h(n)-a-1}) \restriction s$. Note that s is a testing stage for position $n-1$, so it holds that $s > h(b)+a+1 \geq h(n)+a+1$ and $\gamma_s + 2^{-h(n)-a-1} \geq \lfloor \gamma_s \rfloor_{h(n)} + 2^{-h(n)}$. Thus,

$$\gamma_t \geq \left\lfloor \gamma_s + 2^{-h(n)-a-1} \right\rfloor_s \geq \lfloor \gamma_s \rfloor_s + \left\lfloor 2^{-h(n)-a-1} \right\rfloor_s = \gamma_s + 2^{-h(n)-a-1} \geq \lfloor \gamma_s \rfloor_{h(n)} + 2^{-h(n)}. \quad \square$$

Claim 3.11. *If t is a loading stage for position $n-1$ and $\mathcal{Q}$ requires attention again at a later stage s, then the following cases hold.*

(i) If r_{t-1} is even, then $\gamma_s - \gamma_t > 2^{-h(n)-a-1}$.
(ii) If r_{t-1} is odd, supposing s' is the latest testing stage before t, then

$$\gamma_s - \gamma_{s'} > 2^{-h(n)} \geq 2^{a+1} \cdot (\mu(E_{a,s'}) - \mu(E_{a,s'-1})). \tag{6}$$

Proof. First let t' be the least stage in $(t,s]$ such that $\mathcal{Q}$ requires attention. Then by Lemma 2.2 we have

$$\gamma_s \geq \gamma_{t'} \geq \lfloor \gamma_t \rfloor_{h(n)} + 2^{-h(n)}. \tag{7}$$

(i): As r_{t-1} is even and t is a loading stage for position $n-1$, by construction it must hold that $\gamma_t + 2^{-h(n)-a-1} < \lfloor \gamma_t \rfloor_{h(n)} + 2^{-h(n)}$. With (7), it implies $\gamma_s > \gamma_t + 2^{-h(n)-a-1}$.
(ii): In this case, by construction, s' must be a testing stage for position $n-1$. by Claim 3.10, we have

$$\mu(E_{a,s'}) - \mu(E_{a,s'-1}) \leq 2^{-h(n)-a-1}, \tag{8}$$

$$\gamma_t \geq \lfloor \gamma_{s'} \rfloor_{h(n)} + 2^{-h(n)}. \tag{9}$$

(7) and (9) imply $\gamma_s > \gamma_{s'} + 2^{-h(n)}$. With (8), it implies (6). □

Claim 3.12. $\mu(E_a) \leq 2^{-a}$.

Proof. Let $s_0 = 0$ and $0 < s_1 < s_2 < \cdots < s_f$ be all the testing stages. For each $1 \leq i < f$, by Claim 3.8, there is a least loading stage within stages (s_i, s_{i+1}). Then by (ii) of Claim 3.11,

$$\gamma_{s_{i+1}} - \gamma_{s_i} > 2^{a+1} \cdot (\mu(E_{a,s_i}) - \mu(E_{a,s_i-1})). \tag{10}$$

Note that $E_{a,s_i-1} = E_{a,s_{i-1}}$ and $E_{a,s_0} = \emptyset$, so (10) implies

$$\mu(E_{a,s_{f-1}}) < 2^{-a-1} \cdot (\gamma_{s_f} - \gamma_{s_1}) < 2^{-a-1}. \tag{11}$$

On the other hand, suppose s_f is a testing stage for position $n-1$. Then by Claim 3.10, we have

$$\mu(E_{a,s_f}) - \mu(E_{a,s_f-1}) \leq 2^{-h(n)-a-1} \leq 2^{-a-1} \tag{12}$$

As $E_{a,s_f-1} = E_{a,s_{f-1}}$ and $E_a = E_{a,s_f}$, (11) and (12) imply that $\mu(E_a) \leq 2^{-a}$. □

Claim 3.13. *If at some stage s, $r_s = 2q$ then $\mathcal{Q}$ will never require attention at any later stage.*

Proof. For a contradiction, suppose for some s we have $r_s = 2q$ and $\mathcal{Q}$ requires attention at some stage $s' > s$. Then there are already q many loading stages before stage s'. Let $t_0 = 0$ and $0 < t_1 < t_2 < \cdots < t_q$ be these loading stages.

For each $1 \leq i \leq q$, if r_{t_i-1} is even, let $s_i = t_i$; if r_{t_i-1} is odd, there must be a testing stage t' within stages (t_{i-1}, t_i), let $s_i = t'$. Let $s_{q+1} = s'$. Then for each $1 \leq i \leq q$ it holds that $s_i \leq t_i < s_{i+1}$.

Now fix some $1 \leq i \leq q$. Suppose t_i is a loading stage for position n. Note that $\mathcal{Q}$ requires attention at stages s_i, t_i, s_{i+1} respectively (s_i and t_i may be identical). If $s_i = t_i$, then r_{t_i-1} is even. By (i) of Claim 3.11, we have $\gamma_{s_{i+1}} - \gamma_{s_i} > 2^{-h(n+1)-a-1}$. If $s_i < t_i$, then r_{t_i-1} is odd. By (ii) of Claim 3.11, we also have $\gamma_{s_{i+1}} - \gamma_{s_i} > 2^{-h(n+1)} > 2^{-h(n+1)-a-1}$. On the other hand, it is easy to check that $(0, t_q]$ is an α_a $[a, b)$-loading process with loading stages $\{t_i\}_{1 \leq i \leq q}$. By Lemma 3.2 for each $n \in [a, b)$, there are 2^{n-a} many loading stages where the least changed position of α_a is n. Thus,

$$\gamma_{s_{q+1}} - \gamma_{s_1} > \sum_{a \leq n < b} 2^{n-a} \cdot 2^{-h(n+1)-a-1} = 2^{-2a-2} \cdot \sum_{a < n \leq b} 2^{-g(n)} \geq 1.$$

Then $\gamma \geq \gamma_{s_{q+1}} > 1$, which contradicts to our assumption that $\gamma < 1$. □

By Claim 3.13, actually $r_{s-1} < 2q$ already holds when $\mathcal{Q}$ requires attention at stage s. Thus, by construction there are only finitely many stages where $\mathcal{Q}$ requires attention, which means that $\mathcal{Q}$ will be satisfied eventually. This completes the proof of Lemma 3.6. □

Now we prove Theorem 1.5. Let $\{\langle \Phi_i, \gamma_i \rangle\}_{i \in \omega}$ be an effective enumeration of all pairs of a Turing functional with use function bounded by $\mathrm{id} + g$ and a left-c.e. real in $[0, 1)$. Let ℓ be the function as given by Lemma 3.6. Fix a computable series $\{a_e\}_{e \in \omega}$ of natural numbers such that $a_0 = 0$ and $a_{e+1} = a_e + \ell(a_e) + 2$ for all $e \geq 0$. Then uniformly for each $e = \langle i, j \rangle \geq 0$, by Lemma 3.6, there are a left-c.e. real $\alpha_e \subseteq [a_e, a_e + \ell(a_e)]$ and a c.e. set E_e of strings with $\mu(E_e) < 2^{-a_e}$ such that

$$\exists\, x \in [a_e, a_e + \ell(a_e)]\; \alpha_e(x) \neq \Phi_i^{\gamma_i}(x) \quad \text{or} \quad \gamma_i \in [\![E_e]\!]. \tag{13}$$

Now let $\alpha = \sum_{e \in \omega} \alpha_e$. We check that it is as required. First, since α_e are uniformly left-c.e. in e, α is a left-c.e. real. Then fix some $i \geq 0$. Suppose $\alpha = \Phi_i^{\gamma_i}$. It suffices to show that γ_i is not random. Since $\alpha_e \subseteq [a_e, a_e + \ell(a_e)]$, and $a_e + \ell(a_e) < a_{e+1}$, we have

$$\alpha \upharpoonright [a_e, a_e + \ell(a_e)] = \alpha_e \upharpoonright [a_e, a_e + \ell(a_e)]. \tag{14}$$

Thus, with (13) it implies that for each $j \geq 0$, $\gamma_i \in [\![E_{\langle i,j \rangle}]\!]$. Let

$$U_n = \bigcup_{j > n} [\![E_{\langle i,j \rangle}]\!].$$

Then $\gamma_i \in U_n$ for all $n \geq 0$. Note that by our choice of $\{a_e\}_{e \in \omega}$, we have $a_e \geq e$ for all $e \geq 0$. Then for all $j \geq 0$,

$$\mu(E_{\langle i,j\rangle}) < 2^{-a_{\langle i,j\rangle}} \le 2^{-\langle i,j\rangle} \le 2^{-j},$$

where we have the last inequality by choosing a pairing function such that $\langle i, j\rangle \ge j$. Thus,

$$\mu(U_n) \le \sum_{j>n} \mu(E_{\langle i,j\rangle}) < \sum_{j>n} 2^{-j} \le 2^{-n}.$$

On the other hand, U_n are uniformly c.e. classes. Therefore, $\{U_n\}_{n\in\omega}$ is a Martin-Löf test containing γ_i, i.e., γ_i is not random. This proves the first part of the theorem.

To prove such a left-c.e. real exists in every array noncomputable c.e. degree, let $\mathbf{d}$ be an array noncomputable c.e. degree. Fix a very strong array $\{F_n\}_{n\in\omega}$ such that

$$|F_n| \ge \max_{i\le n} 2^{\ell(a_{\langle i,n\rangle})+1}. \tag{15}$$

By Proposition 2.4, let D be an $\{F_n\}_{n\in\omega}$-a.n.c c.e. set in $\mathbf{d}$.

Along with a c.e. set V_i for each $i \ge 0$, uniformly for each $e = \langle i, j\rangle \ge 0$, we effectively transform α_e into a left-c.e. real $\widehat{\alpha}_e$ as follows:

- If $i > j$, let $\widehat{\alpha}_e = 0$ and $V_i \cap F_j = \emptyset$.
- If $i \le j$, we do the following.

Phase 0: Let $\widehat{\alpha}_{e,0} = \alpha_{e,0}$ and $V_{i,0} \cap F_j = \emptyset$. Then go to Phase 1.
Phase 1: Search for a later stage s where $\alpha_{e,s+1} \ne \alpha_{e,s}$. If exists, find some $x \in F_j \setminus V_{i,s}$ and let $V_{i,s+1} = V_{i,s} \cup \{x\}$ and then go to Phase 2.
Phase 2: Search for a later stage t where $D_t \cap F_j = V_{i,t} \cap F_j$. If exists, let $\widehat{\alpha}_{e,t} = \alpha_{e,t}$ and go to Phase 1.

Claim 3.14. *The construction is well defined. Moreover, for $i \le j$, the construction of $\widehat{\alpha}_e$ stays in either Phase 1 or Phase 2 forever after finitely many stages.*

Proof. This is guaranteed by our choice of the very strong array. Suppose $i \le j$. Note that α_e is left-c.e. and $\alpha_{e,s} \subseteq [a_e, a_e + \ell(a_e)]$ for all s, so there are at most $2^{\ell(a_{\langle i,j\rangle})+1} - 1$ many stages where $\alpha_{e,s+1} \ne \alpha_{e,s}$. While by (15), $|F_j| \ge 2^{\ell(a_{\langle i,j\rangle})+1}$. Thus, in Phase 1 whenever α_e changes, the required x can always be found. Moreover, it also implies that the construction of $\widehat{\alpha}_e$ only changes from Phase 1 to Phase 2 for finitely many times. Thus, after finitely many stages, it will stay forever in either Phase 1 or Phase 2. □

Claim 3.15.

(i) For each i, V_i is a c.e. set.
(ii) $\widehat{\alpha}_e \subseteq [a_e, a_e + \ell(a_e)]$ and they are uniformly left-c.e. reals.
(iii) $\widehat{\alpha}_e$ are computable from D.

Proof. (i) and (ii) are obvious by construction.
(iii): If $i > j$, it is trivial. Now suppose $i \le j$. Suppose $\widehat{\alpha}_e$ changed at stage t, i.e., $\widehat{\alpha}_{e,t} \ne \widehat{\alpha}_{e,t-1}$. By construction, this stage t must be at the end of Phase 2, which means that $D_t \cap F_j = V_{i,t} \cap F_j$ and $D_{t-1} \cap F_j \ne V_{i,t-1} \cap F_j$. As stage t is on Phase 2, and $V_i \cap F_j$ doesn't change on Phase 2, it implies $D_{t-1} \cap F_j \ne D_t \cap F_j$, i.e., $D \cap F_j$ also changed at stage t. Thus, to compute $\widehat{\alpha}_e$, we find a stage s such that $D_s \cap F_j = D \cap F_j$. Then $\widehat{\alpha}_e = \widehat{\alpha}_{e,s}$. □

Claim 3.16. *For each $i \ge 0$, there are infinitely many j such that $\widehat{\alpha}_{\langle i,j\rangle} = \alpha_{\langle i,j\rangle}$.*

Proof. As V_i is a c.e. set and D is array noncomputable with respect to $\{F_j\}_{j\in\omega}$, there are infinitely many j such that $V_i \cap F_j = D \cap F_j$. Fix one such j. Note that the construction of $\widehat{\alpha}_{\langle i,j\rangle}$ cannot stay in Phase 2 forever. Then by Claim 3.14, the construction of $\widehat{\alpha}_{\langle i,j\rangle}$ stays in Phase 1 forever after finitely many stages. It enters into this last Phase either from Phase 0 or from Phase 2. In either case, it means that there is a stage t, $\widehat{\alpha}_{\langle i,j\rangle,t} = \alpha_{\langle i,j\rangle,t}$ and $\alpha_{\langle i,j\rangle,s+1} = \alpha_{\langle i,j\rangle,s}$ for all $s \ge t$. Thus, $\widehat{\alpha}_{\langle i,j\rangle} = \widehat{\alpha}_{\langle i,j\rangle,t} = \alpha_{\langle i,j\rangle,t} = \alpha_{\langle i,j\rangle}$. □

Now let

$$\widehat{\alpha} = \sum_{e\in\omega} \widehat{\alpha}_e + \sum_{e\in\omega} D(e)\cdot 2^{-b_e-1},$$

where $b_e = a_e + \ell(a_e) + 1$. First, since $\widehat{\alpha}_e$ are uniformly left-c.e. in e and D is a c.e. set, $\widehat{\alpha}$ is a left-c.e. real.

Second, by Claim 3.15.(iii), $\widehat{\alpha} \leq_T D$. On the other hand, $D(e) = \widehat{\alpha}(b_e)$, i.e., $D \leq_T \widehat{\alpha}$. Thus, $\widehat{\alpha} \equiv_T D \in \mathbf{d}$.

Finally, fix some $i \geq 0$. Suppose $\widehat{\alpha} = \Phi_i^{\gamma_i}$. By Claim 3.16, there are infinitely many j such that $\widehat{\alpha}_{\langle i,j\rangle} = \alpha_{\langle i,j\rangle}$. With (13) and (14), it implies that for infinitely many j, $\gamma_i \in [\![E_{\langle i,j\rangle}]\!]$. The same as before, let $U_n = \bigcup_{j>n} [\![E_{\langle i,j\rangle}]\!]$. It still holds that $\gamma_i \in U_n$ for all $n \geq 0$. Thus, γ_i is not random. This completes the proof.

4. Discussion

Theorem 1.4 and Theorem 1.5 are examples showing that theorems about cl-reducibility can be extended to (id+g)-bT-reducibility where g is a computable slow-growing function. However, there are examples where such extensions fail.

Recall that a set A is *complex* if there is a computable nondecreasing unbounded function h such that $K(A \upharpoonright n) \geq h(n)$ for almost all n or, equivalently, if there is a computable function f such that $K(A \upharpoonright f(n)) \geq n$ for almost all n. Fan and Yu [12] showed that there is a left-c.e. real not cl-reducible to any left-c.e. complex real. However, in a personal communication, Klaus Ambos-Spies demonstrated that this result does not extend to (id+g)-bT-reducibility for any computable nondecreasing unbounded function g.

Theorem 4.1 *(Klaus Ambos-Spies). Let the function $g\colon \mathbb{N} \mapsto \mathbb{N}$ be computable, nondecreasing, and unbounded. Then every left-c.e. real is* (id+g)*-bT-reducible to some left-c.e. complex real.*

Proof. Let $c_n = n + g(n+1)$, let $C = \{c_0, c_1, \ldots\}$ and let $D = \mathbb{N} \setminus C$. Choose $d_0 < d_1 < \cdots$ such that $D = \{d_0, d_1, \ldots\}$. Obviously, both sets C and D are computable. It is easy to see that there are $n+1$ elements in $C \cap [0, n+g(n+1)]$, and thus, $g(n+1)$ elements in $D \cap [0, n+g(n+1)]$. As g is nondecreasing and unbounded, both sets C and D are infinite.

Given a left-c.e. real α we construct a left-c.e. real β as follows. Fix left-c.e. approximations $\{\alpha_s\}_{s\in\omega}$ of α and $\{\Omega_s\}_{s\in\omega}$ of Chaitin's Ω. At stage s, let β_s be the dyadic rational of the form $0.\beta_s(0)\cdots\beta_s(s-1)$ such that for all $c_n, d_n < s$, we have

$$\beta_s(c_n) = \alpha_s(n) \quad \text{and} \quad \beta_s(d_n) = \Omega_s(n). \tag{16}$$

It is easy to check that the β_s converges to some left-c.e. real β that satisfies (16) for all n with all subscripts s removed. Thus, $\alpha(n-1) = \beta(n-1+g(n))$ for all $n \geq 1$, which implies that

$$\alpha \upharpoonright n \text{ is computable from } \beta \upharpoonright n + g(n) \text{ for all } n \geq 0. \tag{17}$$

Thus, α is (id+g)-bT-reducible to β. By construction, (16) also implies

$$\Omega \upharpoonright g(n) \text{ is computable from } \beta \upharpoonright n + g(n) \text{ for all } n \geq 0. \tag{18}$$

Therefore, we have $K(\beta \upharpoonright n + g(n)) \geq K(\Omega \upharpoonright g(n)) - c$ for some constant c and all n. Recall that Ω is Martin-Löf random and thus $K(\Omega \upharpoonright n) \geq n + c$ for every constant c and almost all n. Consequently, we have $K(\beta \upharpoonright n + g(n)) \geq g(n)$ for almost all n, hence β is complex as can be seen by letting $f(n) = i + g(i)$ for the least i where $g(i) \geq n$. $\square$

Theorems 1.4 and 1.5 assert both that for any computable slow-growing function g, there exists an instance, i.e., either a pair of left-c.e. reals or a single left-c.e. real, that has some property that depends on g. A natural question to ask is whether these results can be strengthened by switching quantifiers. More specifically, one may ask the following questions.

Question 1. *Is there a pair of left-c.e. reals such that no left-c.e. real* (id+g)*-bT-computes both of them for any computable slow-growing function g? In case the answer is yes, which degrees contain such pairs?*

Question 2. *Is there a left-c.e. real that is not* (id+g)*-bT-reducible to any left-c.e. random real for any computable slow-growing function g? In case the answer is yes, which degrees contain such reals?*

Apparently, both questions cannot be answered using the techniques presented in this paper. We conjecture that both questions have positive answers, and that the asserted instances can be found exactly in the not totally ω-c.e. degrees. However, these conjectures require additional tools and further investigation, which we will leave for future work.

By Theorem 1.7, for any function g that is computable and fast-growing, every left-c.e. random real (id+g)-bT-computes all other left-c.e. reals, i.e., in a setting of left-c.e. reals, all Martin-Löf random reals have the same computation power when restricted to use (id+g). A similar assertion does not hold for slow-growing functions in general. Barmpalias and Lewis-Pye [2] proved the following theorem.

Theorem 4.2 *(Barmpalias and Lewis-Pye [2]). Given any left-c.e. random real α there exists another left-c.e. random real which is not* (id+ log)*-bT-reducible to α.*

So we know that under the computation with use restricted to id+ log, there are two left-c.e. random reals that are not computationally equivalent, whereas the general case of an arbitrary slow-growing function is open.

Question 3. *Given any computable slow-growing function g, are there always two left-c.e. random reals such that one is not* $(\mathrm{id}+g)$-*bT-reducible to the other?*

CRediT authorship contribution statement

Nan Fang: Writing – review & editing, Writing – original draft, Project administration, Methodology, Investigation. **Wolfgang Merkle:** Writing – review & editing, Supervision, Investigation, Methodology.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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