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Integer-valued martingales and cl-Turing reductions ☆

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ABSTRACT

We prove that a sequence X is integer-valued random (IVR) if and only if for every cl-tt reduction Φ, there are only finitely many oracles that compute X via Φ. A well-known property of integer-valued supermartingales is that under any cone, there is a subcone where the values are constant. We extend this property and prove that for a special integer-valued martingale M, which starts with initial capital 1 and always bets 1 on bit "1" if possible, and for any integer-valued supermartingale f, under any cone there is a subcone such that f actually simulates M. Using this property, we prove that there exists a non-IVR sequence that is cl-computable by only countably many oracles, as a complement to the first result.

1. Introduction

In algorithmic randomness, a *supermartingale* is a function $f : 2^{<\omega} \to \mathbb{Q}^{\geq 0}$ which satisfies the fairness property, i.e.,

$$f(\sigma^\frown 0) + f(\sigma^\frown 1) \leq 2f(\sigma) \quad \text{for all } \sigma \in 2^{<\omega}.$$

It is a *martingale* if only the equality applies. As one approach to defining algorithmic randomness, we say an infinite binary sequence X is random if there is no effective supermartingale that *succeeds* on it. That is, fixing some class of effective supermartingales, for any supermartingale f in this class,

$$\limsup_{n\to\infty} f(X \upharpoonright n) < \infty.$$

Here $X \upharpoonright n$ is the initial segment of X up to length n. Clearly, different classes of supermartingales could result in different notions of randomness. The most studied notion of Martin-Löf randomness is defined by the class of left-c.e. supermartingales, where a function f is *left-c.e.* if there is a uniformly computable family $\{f_s\}_{s\in\omega}$ of functions such that for all $\sigma \in 2^{<\omega}$,

$$f(\sigma) = \lim_s f_s(\sigma) \quad \text{and} \quad f_s(\sigma) \leq f_{s+1}(\sigma) \text{ for all } s \geq 0.$$

More details and background on this can be found in the monograph by Downey and Hirschfeldt [4].

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In real-world casinos, the stakes cannot be arbitrarily small. Usually, there is a minimum unit value, and all capitals are multiples of this unit value. Thus, it makes sense to consider martingales that only take integer values, as well as the induced randomness notion.

Definition 1.1 *(Bienvenu, Stephan, and Teutsch [3]).*

(i) A (super)martingale is called *integer-valued* if it only takes integer values.
(ii) An infinite binary sequence X is *integer-valued random (IVR)* if no integer-valued computable martingale succeeds on it.

In this paper, we explore further properties of integer-valued (super)martingales and reveal their correspondence with cl-reducibilities. Before we go further, let us fix some basic notation and conventions.

Through out this paper, the term *string* always refers to a finite binary string. We denote the set of finite binary strings by $2^{<\omega}$, the set of strings of length d by 2^d, the set of strings of length greater than or equal to d by $2^{\geq d}$, and the empty string by λ. For a string σ, its *length* is denoted by $|\sigma|$. If $\sigma \neq \lambda$, let σ^- denote the string obtained by removing the last bit of σ. Let $[\sigma]$ denote the set of all finite strings that extend σ and $[\![\sigma]\!]$ denote the set of all infinite binary sequences that extend σ. Given two strings σ and τ, we write $\sigma \prec \tau$ or $\tau \succ \sigma$, if σ is a proper prefix of τ. The corresponding nonstrict versions are denoted by $\preceq$ and $\succeq$. The concatenation of σ and τ is denoted by $\sigma^\frown\tau$. The length-lexicographic order on $2^{<\omega}$ is defined by saying that σ is less than τ (written as $\sigma <_{\mathrm{L}} \tau$) if either $|\sigma| < |\tau|$ or else both $|\sigma| = |\tau|$ and $\sigma(n) = 0$ for the least number n such that $\sigma(n) \neq \tau(n)$.

Our first observation is the following proposition.

Proposition 1.2.

(i) An integer-valued computable supermartingale is bounded above by an integer-valued computable martingale.
(ii) Every integer-valued left-c.e. martingale is actually computable.

Proof. "(i):" This is obvious.
"(ii):" Given an integer-valued left-c.e. martingale f, suppose the initial capital of f is c and $\{f_i\}_{i\in\omega}$ is a uniformly computable family of integer-valued supermartingales that converge to f from below. As f is a martingale, for each $k \geq 0$, there must be a stage s such that $\sum_{\sigma\in 2^k} f_s(\sigma) = c \cdot 2^k$. Let $h(k)$ be the least such stage. Then h is a well-defined computable function. Clearly, $f(\sigma) = f_{h(|\sigma|)}(\sigma)$. Thus, f is computable. $\square$

By Proposition 1.2, integer-valued computable (super)martingales and integer-valued left-c.e. martingales have the same power, while according to the following theorem, integer-valued partial computable martingales, as well as integer-valued left-c.e. supermartingales, are strictly more powerful.

Theorem 1.3 *(Barmpalias, Downey, and McInerney [1]). There exists an integer-valued random sequence that is not partial integer-valued random.*

Here, a sequence is partial integer-valued random if no integer-valued partial computable martingale succeeds on it. Note that any partial computable martingale d can be turned into a left-c.e. supermartingale $\hat{d}$ by defining $\hat{d}_0(\sigma)$ to be 0 for all σ and $\hat{d}_s(\sigma) = d_s(\sigma)$ whenever $d_s(\sigma)$ is defined for $s > 0$. Thus, this also implies the following proposition.

Proposition 1.4. *There exists an integer-valued left-c.e. supermartingale that is not computable.*

Definition 1.5. A Turing functional Φ is d-cl for some integer $d \geq 0$, if for all $\sigma \in 2^{\geq d+1}$, either $\Phi^\sigma(|\sigma| - d - 1)\downarrow$, or $\Phi^\rho(n)\uparrow$ for all $\rho \succeq \sigma$ and $n \geq |\sigma| - d - 1$. It is d-cl-tt if only the first case applies. It is cl (cl-tt) if it is d-cl (d-cl-tt) for some integer d.

Given a d-cl Turing functional Φ, we can see it as a map from $2^{\geq d}$ to $2^{<\omega}$.

Definition 1.6. Let T be a partial map from $2^{\geq d}$ to $2^{<\omega}$. We say T is a d-cl-map if all of the following hold.

i) For all $\sigma \in 2^d$, either $T(\sigma) = \lambda$ or $T(\sigma)\uparrow$.
ii) For all $\sigma \in 2^{>d}$, if $T(\sigma)$ is defined, then $T(\sigma^-)$ is defined and $T(\sigma)^- = T(\sigma^-)$.

We say T is a cl-map if it is a d-cl-map for some $d \geq 0$.

Clearly, for any d-cl-reduction Φ, by letting $T_\Phi(\sigma) = \tau$ whenever $\sigma \in 2^{|\tau|+d}$ and $\Phi^\sigma \upharpoonright |\tau| = \tau$, we get a d-cl-map T_Φ. We say T_Φ is the corresponding cl-map for Φ. For any sequence $X \in 2^\omega$, let $T_\Phi^{-1}(X)$ be the set of oracles that compute X by Φ. It is easy to check that T_Φ is partial computable and it is total when Φ is a cl-tt-reduction. Thus, the following proposition holds.

Proposition 1.7. *For any cl-reduction Φ, $|T_\Phi^{-1}(\cdot)|$ is an integer-valued left-c.e. supermartingale. It is a computable martingale in case Φ is total. And for any X, $|T^{-1}(X)| \le \lim_n |T^{-1}(X \upharpoonright n)|$.*

This proposition has already been observed and used by Lewis and Barmpalias [5]. As a generalization of integer-valued martingales, Barmpalias, Lewis-Pye, and Teutsch [2] defined the notion of *granular-martingale*. Fixing a nondecreasing computable function $g : \mathbb{N} \to \mathbb{N}$, roughly speaking, a martingale M is g-*granular* if for every string σ, the value of $M(\sigma)$ is an integer multiple of $2^{-g(|\sigma|)}$. One can also extend the notion of cl-reduction to g-*redundant reduction*. A Turing reduction is called g-*redundant* if its use function is bounded by $n + g(n)$. In the same spirit as Proposition 1.7, Barmpalias, Lewis-Pye, and Teutsch [2] showed that one can construct g-granular left-c.e. supermartingales from g-redundant Turing functionals.

One main purpose of this paper is to demonstrate the other direction. In Section 3, we construct a corresponding cl-tt-reduction from every integer-valued martingale. Thus, it implies that a sequence X is IVR if and only if for every cl-tt reduction Φ, there are only finitely many oracles that compute X via Φ (Theorem 3.1). This also implies that every IVR sequence can only be cl-tt-computed by countably many oracles. However, the converse is not true. We also prove that there exists a non-IVR sequence that is cl-computed by only countably many oracles (Theorem 3.3).

As a byproduct of the later proof, we reveal an interesting property of integer-valued supermartingales, i.e., there is a special integer-valued martingale M such that for any integer-valued supermartingale f, under any cone there is a subcone where f actually simulates M. This property can be seen as a generalization of a well-known property of integer-valued supermartingales, which states that for an integer-valued supermartingale, under any of its local cones, there is a subcone where the values are constant. As this property is interesting in itself and might be a useful tool in dealing with integer-valued supermartingales, we present it in Section 2.

With Proposition 1.4, it is worth noting that Theorem 3.1 is for cl-tt-reductions while Theorem 3.3 is for cl-reductions. Therefore, we would like to ask the following question.

Question 1. *Given an integer-valued left-c.e. supermartingale f, can we find a constant c and a cl-reduction Φ such that $|T_\Phi^{-1}(\sigma)| = f(\sigma) + c$ for all $\sigma \in 2^{<\omega}$, and such that $\lim_n f(X \upharpoonright n) = \infty$ implies $|T_\Phi^{-1}(X)| = \infty$ for all X?*

2. Property of integer-valued martingales

The following lemma implicitly from [4, Lemma 9.13.1] and also from [3, Lemma 4] states a useful and well-known property for integer-valued supermartingales. We include its proof here as an example for the basic proof ideas of the two lemmas that follow.

Lemma 2.1 *([4]; [3]). Given any integer-valued supermartingale f, for any string σ_0, there exist a string $\sigma \succeq \sigma_0$ and a constant $r \le f(\sigma_0)$ such that $f(\tau) = r$ for all $\tau \succeq \sigma$.*

Proof. Let $L = \{\tau : \tau \succeq \sigma_0\}$. Let σ be the least string in L under the length-lexicographic order such that $f(\sigma) = \min_{\tau \in L} f(\tau)$. Let $r = f(\sigma)$. As $\sigma_0 \in L$, it holds that $r \le f(\sigma_0)$. Fix some $\tau \succeq \sigma$ such that $f(\tau) = r$. By the choice of σ, we have

$$f(\tau^\frown 1) \ge r \quad \text{and} \quad f(\tau^\frown 0) \ge r. \tag{1}$$

On the other hand, as f is a supermartingale, we have

$$f(\tau^\frown 1) + f(\tau^\frown 0) \le 2f(\tau) = 2r. \tag{2}$$

Therefore, by (1) and (2) we have $f(\tau^\frown 1) = f(\tau^\frown 0) = r$. By induction, it holds that $f(\tau) = r$ for all $\tau \succeq \sigma$. $\square$

Let M be a computable integer-valued martingale defined as follows. It starts with initial capital 1 and always bets 1 on "1" if possible. That is,

i) $M(\lambda) = 1$.
ii) For any $\sigma \in 2^{<\omega}$ and $i \in \{0, 1\}$,

$$M(\sigma^\frown i) = \begin{cases} M(\sigma) + 1 & \text{if } M(\sigma) \ge 1 \text{ and } i = 1; \\ M(\sigma) - 1 & \text{if } M(\sigma) \ge 1 \text{ and } i = 0; \\ M(\sigma) & \text{if } M(\sigma) = 0. \end{cases}$$

For $c < M(\sigma)$, let $[\sigma : M > c]$ be the collection of extensions of σ, where starting from σ all along the extension the values of M are always above c. That is,

$$[\sigma : M > c] = \{\tau \succeq \sigma : \forall \sigma \preceq \eta \preceq \tau \ (M(\eta) > c)\}.$$

It's easy to check that for this notation the following property holds.

$$\text{If } \tau \in [\sigma : M > c] \text{ and } d \ge c, \text{ then } [\tau : M > d] \subseteq [\sigma : M > c]. \tag{3}$$

As an extension to Lemma 2.1, the next two lemmas show that for any integer-valued martingale f, under every local cone, there is a subcone such that f actually simulates M.

Lemma 2.2. *Given any integer-valued supermartingale f, for any string σ_0 with $M(\sigma_0) > f(\sigma_0)$, there exists a string $\sigma \in \left[\sigma_0 :\ M > 0\right]$ and a constant $r \le f(\sigma_0)$ such that $r < M(\sigma)$ and $f(\tau) = r$ for all $\tau \in [\sigma :\ M > r]$.*

Proof. Let $L = \{\tau \in \left[\sigma_0 :\ M > 0\right] \,:\, f(\tau) < M(\tau)\}$. As $\sigma_0 \in L$, L is not empty. Under the length-lexicographic order, let σ be the least string in L such that $f(\sigma) = \min_{\tau\in L} f(\tau)$. As f is integer-valued and nonnegative, such σ exists. Let $r = f(\sigma)$. Then $r < M(\sigma)$. Now we verify that σ and r are as required. First, note that $f(\sigma) \le f(\sigma_0)$. So $r \le f(\sigma_0)$. Then fix some $\tau \in [\sigma :\ M > r]$ such that $f(\tau) = r$.

- Suppose $f(\tau^\frown 1) < r$. Then

  $$0 \le f(\tau^\frown 1) < f(\tau) < M(\tau) < M(\tau^\frown 1).$$

  Thus $\tau^\frown 1 \in L$, and $f(\tau^\frown 1) < f(\sigma)$, which contradicts the choice of σ. Therefore,

  $$f(\tau^\frown 1) \ge r. \tag{4}$$

- Suppose $f(\tau^\frown 0) < r$. Then

  $$0 \le f(\tau^\frown 0) \le f(\tau) - 1 < M(\tau) - 1 = M(\tau^\frown 0).$$

  Thus $\tau^\frown 0 \in L$, and $f(\tau^\frown 0) < f(\sigma)$, which contradicts the choice of σ as well. Therefore,

  $$f(\tau^\frown 0) \ge r. \tag{5}$$

- On the other hand, as f is a supermartingale, we have

  $$f(\tau^\frown 1) + f(\tau^\frown 0) \le 2f(\tau) = 2r. \tag{6}$$

Therefore, by (4), (5) and (6) we have $f(\tau^\frown 1) = f(\tau^\frown 0) = r$. By induction, it holds that $f(\tau) = r$ for all $\tau \in [\sigma :\ M > r]$. $\square$

The lemma above can be further generalized, leading to the following result. Note that the earlier lemma is a special case of this general form, where $q = q_0 = 0$.

Lemma 2.3. *Given any integer-valued supermartingale f, for any string σ_0 with $M(\sigma_0) > 0$, there exist a string $\sigma \in \left[\sigma_0 :\ M > 0\right]$ and constants $q, r \in \mathbb{N}$ such that $r < M(\sigma)$ and $f(\tau) = qM(\tau) + r$ for all $\tau \in [\sigma :\ M > r]$. Moreover, it holds that either $q < q_0$ or $(q = q_0 \wedge r \le r_0)$, where q_0, r_0 are natural numbers such that $0 \le r_0 < M(\sigma_0)$ and $f(\sigma_0) = q_0 M(\sigma_0) + r_0$.*

Proof. Let $L = \left[\sigma_0 :\ M > 0\right]$. For all $\tau \in L$, let q_τ, r_τ be natural numbers such that

$$f(\tau) = q_\tau M(\tau) + r_\tau \quad \text{and} \quad 0 \le r_\tau < M(\tau).$$

We define an order $<_p$ on the pairs (q_τ, r_τ) as follows:

$$(q_1, r_1) <_p (q_2, r_2) \quad \text{iff} \quad q_1 < q_2 \text{ or } (q_1 = q_2 \wedge r_1 < r_2).$$

Under the length-lexicographic order, let σ be the least string in L such that (q_σ, r_σ) achieves the minimal value under the order $<_p$. Note that q_τ is integer-valued and nonnegative for all $\tau \in L$, so it must have a minimal value. Fix some q. For all $\tau \in L$ such that $q_\tau = q$, r_τ is also integer-valued and nonnegative. Thus, such r_τ must also have a minimal value. Therefore, σ is well-defined. Let $q = q_\sigma$ and $r = r_\sigma$.

Now we verify that σ, q, r are as required. Since $\sigma_0 \in L$, we have $(q_\sigma, r_\sigma) \le_p (q_{\sigma_0}, r_{\sigma_0})$, i.e., either $q < q_0$ or $(q = q_0 \wedge r \le r_0)$. Fix some $\tau \in [\sigma :\ M > r]$ such that $(q_\tau, r_\tau) =_p (q, r)$.

- Suppose $f(\tau^\frown 1) < f(\tau) + q$. Since $\tau \in [\sigma :\ M > r] \subseteq \left[\sigma_0 :\ M > 0\right]$ and $M(\tau^\frown 1) > M(\tau) > r_\tau \ge 0$, we have $\tau^\frown 1 \in L$. On the other hand, we have

  $$f(\tau^\frown 1) < qM(\tau) + r + q = qM(\tau^\frown 1) + r,$$

  which implies $(q_{\tau^\frown 1}, r_{\tau^\frown 1}) <_p (q, r)$. This contradicts the choice of σ. Thus,

  $$f(\tau^\frown 1) \ge f(\tau) + q. \tag{7}$$

- Suppose $f(\tau^\frown 0) < f(\tau) - q$. First note that in this case, we must have $M(\tau) > 1$. Otherwise, if $M(\tau) = 1$, then $f(\tau) = q$. With $f(\tau^\frown 0) < f(\tau) - q = 0$, this is not possible. So $M(\tau^\frown 0) = M(\tau) - 1 > 0$. Since $\tau \in \left[\sigma_0 :\ M > 0\right]$, it implies $\tau^\frown 0 \in L$. On the other hand, note

$$f(\tau^\frown 0) \le f(\tau) - q - 1 = qM(\tau) + r - q - 1 = qM(\tau^\frown 0) + r - 1.$$

This implies $(q_{\tau^\frown 0}, r_{\tau^\frown 0}) <_p (q, r)$, which contradicts the choice of σ as well. Thus,

$$f(\tau^\frown 0) \ge f(\tau) - q. \tag{8}$$

- Since f is a supermartingale, we have

$$f(\tau^\frown 1) + f(\tau^\frown 0) \le 2f(\tau). \tag{9}$$

Therefore, by (7), (8), and (9), we have

$$f(\tau^\frown 1) = f(\tau) + q = qM(\tau^\frown 1) + r \quad \text{and} \quad f(\tau^\frown 0) = f(\tau) - q = qM(\tau^\frown 0) + r.$$

If $M(\tau^\frown 0) > r$, then

$$(q_{\tau^\frown 1}, r_{\tau^\frown 1}) =_p (q_{\tau^\frown 0}, r_{\tau^\frown 0}) =_p (q, r).$$

By induction, for all $\tau \in [\sigma : M > r]$, we have $(q_\tau, r_\tau) =_p (q, r)$, i.e., $f(\tau) = qM(\tau) + r$. $\square$

By replacing $M(\sigma)$ with $M(\sigma) - c$ for any integer $c > 0$, from Lemma 2.2 and Lemma 2.3, we can actually derive the following two extended lemmas.

Lemma 2.4. *Given any integer-valued supermartingale f, for any string σ_0 and integer $c > 0$ with $M(\sigma_0) > f(\sigma_0) + c$, there exist a string $\sigma \in \left[\sigma_0 : M > c\right]$ and a constant $r \le f(\sigma_0)$ such that $r < M(\sigma) - c$ and $f(\tau) = r$ for all $\tau \in [\sigma : M > r + c]$.*

Lemma 2.5. *Given any integer-valued supermartingale f, for any string σ_0 and integer $c > 0$ with $M(\sigma_0) > c$, there exist a string $\sigma \in \left[\sigma_0 : M > c\right]$ and constants $q, r \in \mathbb{N}$ such that $r < M(\sigma) - c$ and $f(\tau) = q(M(\tau) - c) + r$ for all $\tau \in [\sigma : M > r + c]$.*

3. Integer-valued randomness and cl-reducibilities

Theorem 3.1. *A sequence X is IVR if and only if for every cl-tt reduction Φ, there are only finitely many oracles that compute X via Φ.*

This theorem directly follows from the following lemma:

Lemma 3.2. *For any integer-valued computable martingale f, there exists a total computable cl-map T and a natural number c such that $|T^{-1}(\sigma)| = f(\sigma) + c$ for all $\sigma \in 2^{<\omega}$. Moreover, for any sequence X, we have $\lim_n f(X \upharpoonright n) = \infty$ if and only if $|T^{-1}(X)| = \infty$.*

Proof. Let f be an integer-valued computable martingale. Let d be the least integer such that $f(\lambda) \le 2^d$. Let $c = 2^d - f(\lambda)$ and $g(\sigma) = f(\sigma) + c$ for all $\sigma \in 2^{<\omega}$. Then g is an integer-valued computable martingale with $g(\lambda) = 2^d$.

We define a computable d-cl-map $T : 2^{\ge d} \to 2^{<\omega}$ that satisfies the following properties for all strings $\sigma \in 2^{<\omega}$:

(i) $|T^{-1}(\sigma)| = g(\sigma)$;
(ii) If $\tau \prec \sigma$ and $g(\rho) \ge g(\tau)$ for all $\tau \prec \rho \preceq \sigma$, then every string in $T^{-1}(\tau)$ has an extension in $T^{-1}(\sigma)$.

Clearly, such a d-cl-map T is total.

Construction. We define T by specifying the pre-image $T^{-1}(\sigma)$ of every $\sigma \in 2^{<\omega}$ inductively.

At stage 0, let $T^{-1}(\lambda) = 2^d$.

At stage $s+1$, for each $\sigma \in 2^s$, assuming for all $\rho \preceq \sigma$, $T^{-1}(\rho)$ are defined and properties (i) and (ii) are satisfied, we define $T^{-1}(\sigma^\frown 0)$ and $T^{-1}(\sigma^\frown 1)$ as follows:

1. Find $i \in \{0, 1\}$ such that $g(\sigma^\frown i) \le g(\sigma^\frown(1 - i))$.
2. Search for the longest prefix σ_0 of σ such that $g(\sigma_0) \le g(\sigma^\frown i)$.

   - If exists, let σ_0 be such prefix. Then for all $\sigma_0 \prec \rho \preceq \sigma$, $g(\rho) > g(\sigma^\frown i) \ge g(\sigma_0)$. By induction assumption, $|T^{-1}(\sigma_0)| = g(\sigma_0)$ and every string in $T^{-1}(\sigma_0)$ has an extension in $T^{-1}(\sigma)$. Let V be a subset of $T^{-1}(\sigma)$ such that $|V| = g(\sigma^\frown i)$ and each string in $T^{-1}(\sigma_0)$ has an extension in V. As $g(\sigma^\frown i) \ge g(\sigma_0)$, such V exists.
   - Otherwise, let σ_0 be undefined, and let V be any subset of $T^{-1}(\sigma)$ such that $|V| = g(\sigma^\frown i)$.

3. Let

$$T^{-1}(\sigma^\frown i) = \bigcup_{\tau \in V} \{\tau^\frown 0\} \quad \text{and} \quad T^{-1}(\sigma^\frown(1 - i)) = \bigcup_{\tau \in T^{-1}(\sigma) \setminus V} \{\tau^\frown 0\} \cup \bigcup_{\tau \in T^{-1}(\sigma)} \{\tau^\frown 1\}.$$

Verification.

Claim 3.2.1. T is well-defined and properties (i) and (ii) are satisfied for all strings $\sigma \in 2^{<\omega}$.

Proof. First, as $T^{-1}(\lambda) = 2^d$, properties (i) and (ii) are satisfied for $\sigma = \lambda$.

Now fix some string $\sigma \in 2^s$. Assume for all $\rho \preceq \sigma$, $T^{-1}(\rho)$ are defined and properties (i) and (ii) are satisfied. By induction it suffices to check that during the construction, at stage $s+1$, the definitions of $T^{-1}(\sigma^\frown 0)$ and $T^{-1}(\sigma^\frown 1)$ satisfy (i) and (ii). Let i, σ_0 and V be as specified during the construction. By construction, it holds that

$$|T^{-1}(\sigma^\frown i)| = |V| = g(\sigma^\frown i)$$

and

$$|T^{-1}(\sigma^\frown(1-i))| = 2|T^{-1}(\sigma)| - |V| = 2g(\sigma) - g(\sigma^\frown i) = g(\sigma^\frown(1-i)).$$

This verifies that property (i) is satisfied for both $\sigma^\frown i$ and $\sigma^\frown(1-i)$.

In case σ_0 is undefined, it implies that $g(\rho) > g(\sigma^\frown i)$ for all $\rho \prec \sigma^\frown i$. Thus, property (ii) is satisfied for $\sigma^\frown i$ trivially. In case σ_0 is defined, recall that σ_0 is the longest prefix of σ such that $g(\sigma_0) \leq g(\sigma^\frown i)$. Suppose $\tau \prec \sigma^\frown i$ and $g(\rho) \geq g(\tau)$ for all $\tau \prec \rho \preceq \sigma^\frown i$. Then it holds that $\tau \preceq \sigma_0$ and $g(\tau) \leq g(\sigma_0)$. Moreover, as $g(\rho) \geq g(\tau)$ for all $\tau \prec \rho \preceq \sigma_0$ and $\sigma_0 \preceq \sigma$, by induction assumption, every string in $T^{-1}(\tau)$ has an extension in $T^{-1}(\sigma_0)$. On the other hand, by construction, every string in $T^{-1}(\sigma_0)$ has an extension in $T^{-1}(\sigma^\frown i)$. Thus, every string in $T^{-1}(\tau)$ has an extension in $T^{-1}(\sigma^\frown i)$. Property (ii) is satisfied for $\sigma^\frown i$ as well.

Now suppose $\tau \prec \sigma^\frown(1-i)$ and $g(\rho) \geq g(\tau)$ for all $\tau \prec \rho \preceq \sigma^\frown(1-i)$. By induction assumption, every string in $T^{-1}(\tau)$ has an extension in $T^{-1}(\sigma)$. Furthermore, by construction, every string in $T^{-1}(\sigma)$ has an extension in $T^{-1}(\sigma^\frown(1-i))$. Thus, property (ii) is satisfied for $\sigma^\frown(1-i)$ as well. □

Now we verify T and c are as required. By property (i), $|T^{-1}(\sigma)| = g(\sigma) = f(\sigma) + c$ for all $\sigma \in 2^{<\omega}$.

Fix any $X \in 2^\omega$. If $|T^{-1}(X)| = \infty$, then

$$\lim_n f(X \upharpoonright n) = \lim_n |T^{-1}(X \upharpoonright n)| - c \geq |T^{-1}(X)| - c = \infty.$$

On the other hand, suppose $\lim_n f(X \upharpoonright n) = \infty$, i.e., $\lim_n g(X \upharpoonright n) = \infty$. Then there exists an increasing sequence of integers $\{n_k\}_{k\in\omega}$ such that for all $k \in \omega$, it holds that

$$\forall n \geq n_k \ \big(g(X \upharpoonright n) \geq g(X \upharpoonright n_k) \geq k\big).$$

Fix some k. For any $n > n_k$, as $g(X \upharpoonright m) \geq g(X \upharpoonright n_k)$ for all $n_k < m \leq n$, by property (ii), each string in $T^{-1}(X \upharpoonright n_k)$ has an extension in $T^{-1}(X \upharpoonright n)$. Then each string in $T^{-1}(X \upharpoonright n_k)$ extends at least one sequence in $T^{-1}(X)$. That is to say, $T^{-1}(X)$ has at least $|T^{-1}(X \upharpoonright n_k)| = g(X \upharpoonright n_k) \geq k$ elements. As $k \to \infty$, we have $|T^{-1}(X)| = \infty$. □

Theorem 3.1 tells us that every IVR sequence can be cl-computed by at most countably many sequences. However, the following theorem tells us that the converse is not true.

Theorem 3.3. *There exists a non-IVR sequence such that only countably many oracles cl-compute it.*

Proof. Let M be the computable integer-valued martingale defined in Section 2. Let $\{\Phi_i\}_{i\in\omega}$ be a list of all cl-Turing functionals and $\{T_i\}_{i\in\omega}$ be the corresponding cl-maps. We construct a sequence X such that $\lim_{n\to\infty} M(X \upharpoonright n) = \infty$, and such that $T_i^{-1}(X)$ is countable for each i. To this end, for each i, we define a tree S_i such that:

(i) S_i has finite width so that it has finitely many paths;
(ii) for each end node $\rho \in S_i$, $T_i^{-1}(X) \cap [\![\rho]\!]$ is finite.

We construct the sequence X by finite extension. We inductively define $\{\sigma_e\}_{e\in\omega}$, $\{m_e\}_{e\in\omega}$ such that the following holds:

$$M(\sigma_e) > m_e, \quad m_{e+1} > m_e, \quad |\sigma_{e+1}| > |\sigma_e|, \quad \text{and} \quad \sigma_{e+1} \in \big[\sigma_e : M > m_e\big]. \tag{10}$$

We also inductively define auxiliary variables $\{c_i\}_{i\in\omega}$, $\{q_i\}_{i\in\omega}$, $\{r_i\}_{i\in\omega}$, and $\{S_{i,j}\}_{i,j\in\omega}$. Let $e = \langle i, j\rangle$, where we choose the pairing function so that $\langle i, j\rangle$ is increasing in both arguments. We also make sure the construction obeys the following rule.

(∗) Each set $S_{i,j}$ only consists of strings of length j. At stage $e+1$, σ_{e+1}, m_{e+1} and $S_{i,j+1}$ are defined, and in case $j = 0$, $S_{i,0}$, c_i, q_i and r_i are also defined with the following property satisfied,

$$|T_i^{-1}(\tau)| = q_i(M(\tau) - c_i) + r_i \quad \text{for all} \quad \tau \in \big[\sigma_{\langle i,0\rangle+1} : M > m_{\langle i,0\rangle+1}\big]. \tag{11}$$

Construction. At stage 0: Let $\sigma_0 = \lambda$ and $m_0 = 0$. Clearly, $M(\sigma_0) = M(\lambda) = 1 > m_0$, so (10) is satisfied.

At stage $e+1 (e \geq 0)$: Let $e = \langle i, j \rangle$. Suppose we have already defined $\{\sigma_k\}_{0 \leq k \leq e}$, $\{m_k\}_{0 \leq k \leq e}$ satisfying (10), and in all previous stages, the construction obeys $(*)$.

- *In case $j = 0$:* Let $c_i = m_e$. As $|T_i^{-1}(\cdot)|$ is an integer-valued supermartingale, by Lemma 2.5, find $\sigma \in \left[\sigma_e : M > c_i\right]$ and $q_i, r_i \in \mathbb{N}$ such that

$$M(\sigma) > r_i + c_i \quad \text{and} \quad |T_i^{-1}(\tau)| = q_i(M(\tau) - c_i) + r_i \text{ for all } \tau \in \left[\sigma : M > r_i + c_i\right].$$

 Let

$$m_{e+1} = c_i + r_i + 1, \quad \sigma_{e+1} = \sigma^\frown 1, \quad S_{i,0} = \{\lambda\}, \quad \text{and} \quad S_{i,1} = \{0, 1\}.$$

 Note that $M(\sigma_{e+1}) = M(\sigma) + 1 > c_i + r_i + 1 = m_{e+1} > m_e$, $|\sigma_{e+1}| > |\sigma| \geq |\sigma_e|$, and $\sigma_{e+1} \in \left[\sigma_e : M > m_e\right]$. Thus σ_{e+1} and m_{e+1} satisfy (10). Moreover, the construction obeys $(*)$ for this stage.
- *In case $j > 0$:* Note that by our choice of pairing function, $e > \langle i, j-1 \rangle \geq \langle i, 0 \rangle$. By induction assumption, c_i, q_i and r_i are already defined at stage $\langle i, 0 \rangle + 1$ and satisfy (11). Moreover, $S_{i,j}$ is also already defined at stage $\langle i, j-1 \rangle + 1 \leq e$.

 Let $L = S_{i,j}$, $\eta = \sigma_e$, and $c = m_e$. If $|L| > q_i$, we repeat the following until $|L| \leq q_i$:

 1. Find the shortest $\eta_0 \in [\eta : M > c]$ such that $M(\eta_0) > q_i(c - c_i) + c + r_i$. Such η_0 can be easily found by appending enough 1's to the end of η. Note that $c \geq m_{\langle i,0 \rangle + 1}$, and $\eta \in \left[\sigma_{\langle i,0 \rangle + 1} : M > m_{\langle i,0 \rangle + 1}\right]$. So $\eta_0 \in [\eta : M > c] \subseteq \left[\sigma_{\langle i,0 \rangle + 1} : M > m_{\langle i,0 \rangle + 1}\right]$. By (11), it holds that
 $$|T_i^{-1}(\eta_0)| = q_i(M(\eta_0) - c_i) + r_i. \tag{12}$$
 2. Find $\rho \in L$ such that $|T_i^{-1}(\eta_0) \cap [\rho]| < M(\eta_0) - c$. Such ρ exists because otherwise there are at least $q_i + 1$ strings in L with $|T_i^{-1}(\eta_0) \cap [\rho]| \geq M(\eta_0) - c$, then
 $$\begin{aligned} |T_i^{-1}(\eta_0)| &\geq \sum_{\rho \in L} |T_i^{-1}(\eta_0) \cap [\rho]| \\ &\geq (q_i + 1)(M(\eta_0) - c) \\ &> q_i(M(\eta_0) - c) + (q_i(c - c_i) + c + r_i) - c \\ &= q_i(M(\eta_0) - c_i) + r_i, \end{aligned}$$
 which contradicts (12). Here the first inequality holds because by induction assumption, all strings in $L \subseteq S_{i,j}$ have the same length j, thus for all strings $\rho \in L$, $[\rho]$ are pairwise disjoint.
 3. We view $|T_i^{-1}(\cdot) \cap [\rho]|$ as an integer-valued supermartingale rooted at η_0. By Lemma 2.4, find $\eta_1 \in \left[\eta_0 : M > c\right]$ and $r \leq |T_i^{-1}(\eta_0) \cap [\rho]|$ such that $M(\eta_1) > c + r$ and
 $$|T_i^{-1}(\tau) \cap [\rho]| = r \quad \text{for all} \quad \tau \in \left[\eta_1 : M > c + r\right]. \tag{13}$$
 4. Remove ρ from L, and let $c + r$ and η_1 be the new values of c and η respectively.

 Note that during the construction, L, c and η are dynamic variables whose values could be updated finitely many times. After there are finalized, let

$$m_{e+1} = c + 1, \quad \sigma_{e+1} = \eta^\frown 1, \quad \text{and} \quad S_{i,j+1} = \{\sigma^\frown 0 : \sigma \in L\} \cup \{\sigma^\frown 1 : \sigma \in L\}.$$

 By construction, we have $\eta \geq \sigma_e$, $c \geq m_e$ and $M(\eta) > c$. It implies $m_{e+1} = c + 1 > m_e$ and $M(\sigma_{e+1}) = M(\eta) + 1 > m_{e+1}$. Also note that $|\sigma_{e+1}| > |\eta| \geq |\sigma_e|$ and $\sigma_{e+1} \in [\eta : M > c] \subseteq \left[\sigma_e : M > m_e\right]$. Thus σ_{e+1} and m_{e+1} satisfy (10). The construction obeys $(*)$ for this stage as well.

Let $X = \lim_{e \to \infty} \sigma_e$. And for each i, let $S_i = \bigcup_{j \geq 0} S_{i,j}$.

Verification. First, our construction guarantees that (10) is satisfied for all e. Thus, for all e, $|\sigma_e| \geq e$ and $M(\sigma_e) > m_e \geq e$. Therefore, X is indeed an infinite sequence, and for all $|\sigma_e| \leq n < |\sigma_{e+1}|$, $M(X \upharpoonright n) > m_e \geq e$. Thus, $\lim_{n \to \infty} M(X \upharpoonright n) = \infty$.

Now fix i. For each j, there are at most q_i many strings of length j that have extensions in S_i. Thus, S_i has at most q_i many paths. On the other hand, for each end node $\rho \in S_i$, by (13), $|T_i^{-1}(X \upharpoonright n) \cap [\rho]|$ is constant for all n large enough, so $|T_i^{-1}(X) \cap [\![\rho]\!]|$ is finite as well. Moreover, by the construction, every sequence either is a path in S_i or has a prefix which is an end node in S_i. Therefore, $T_i^{-1}(X)$ is countable. □

CRediT authorship contribution statement

Nan Fang: Writing – review & editing, Writing – original draft, Methodology, Investigation. **Lu Liu:** Writing – review & editing, Writing – original draft, Methodology, Investigation. **Liang Yu:** Writing – review & editing, Methodology, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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