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Speedability of computably approximable reals and their approximations

George Barmpalias [a], Nan Fang [a,*], Wolfgang Merkle [b], Ivan Titov [b,c]

[a] Key Laboratory of System Software (Chinese Academy of Sciences), Institute of Software, Chinese Academy of Sciences, Beijing, China
[b] Institut für Informatik, Ruprecht-Karls-Universität Heidelberg, Germany
[c] Université de Bordeaux, CNRS, Bordeaux INP, LaBRI, UMR 5800, Talence, F-33400, France

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ABSTRACT

An approximation of a real is a sequence of rational numbers that converges to the real. An approximation is left-c.e. if it is computable and nondecreasing and is d.c.e. if it is computable and has bounded variation. A real is computably approximable if it has some computable approximation, and left-c.e. and d.c.e. reals are defined accordingly.

An approximation $\{a_s\}_{s\in\omega}$ is *speedable* if there exists a nondecreasing computable function f such that the approximation $\{a_{f(s)}\}_{s\in\omega}$ converges in a certain formal sense faster than $\{a_s\}_{s\in\omega}$. This leads to various notions of speedability for reals, e.g., one may require for a computably approximable real that either all or some of its approximations of a specific type are speedable.

Merkle and Titov established the equivalence of several speedability notions for left-c.e. reals that are defined in terms of left-c.e. approximations. We extend these results to d.c.e. reals and d.c.e. approximations, and we prove that in this setting, being speedable is equivalent to not being Martin-Löf random. Finally, we demonstrate that every computably approximable real has a computable approximation that is speedable.

1. Introduction

1.1. Left-c.e. reals and Solovay reducibility

An approximation of a real is a sequence of rational numbers that converges to that real. In what follows, unless explicitly stated otherwise, by real we always refer to a real in $(0,1)$, the open unit interval, and all rational numbers occurring in an approximation are in $[0,1]$, the closed unit interval. An approximation $\{a_s\}_{s\in\omega}$ is *computable* if a_s can be computed for given s. Restricted variants of computable approximations are defined as follows. A computable approximation is *left-c.e.* if it is nondecreasing, *right-c.e.* if it is nonincreasing, and *d.c.e* if it has bounded variation, i.e., if the sum over the terms $|a_s - a_{s+1}|$ is finite. A real is *computably approximable* if it has a computable approximation, a real is left-c.e. if it has a left-c.e. approximation, and right-c.e. and d.c.e. reals are defined accordingly.

Remark 1.1. Every c.a. real has a computable approximation that consists of pairwise distinct dyadic rationals, i.e., of rationals with a denominator of the form 2^k. Likewise, every left-c.e. real has a left-c.e. approximation that consists of pairwise distinct dyadic rationals, and a similar remark holds for right-c.e. and for d.c.e. reals. The easy proofs of these well-known facts are left to the reader.

* Corresponding author.
E-mail addresses: barmpalias@gmail.com (G. Barmpalias), fangnan@ios.ac.cn (N. Fang), merkle@math.uni-heidelberg.de (W. Merkle), me@ivantitov.de (I. Titov).

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Recall that an *order* is a nondecreasing and unbounded function from the set of natural numbers to itself. For further use, note that for any computable order f, given a c.a. approximation $\{a_s\}_{s\in\omega}$, then $\{a_{f(s)}\}_{s\in\omega}$ is a c.a. approximation of the same real. It is easy to check that this implication remains true when replacing the two occurrences of c.a. either both by d.c.e., or both by left-c.e., or both by right-c.e.

The class of left-c.e. reals is a central object of study in the field of algorithmic randomness. Solovay reducibility is used to compare such reals by the speed of convergence of their left-c.e. approximations. The original definition of Solovay reducibility by Solovay [1] applies to all reals, but when restricted to left-c.e. reals, the following equivalent definition due to Calude et al. [2] is better suited and more relevant to our discussion (see also the monograph by Downey and Hirschfeldt [3]). For left-c.e. reals α and β, α is *Solovay reducible* to β, denoted $\alpha \leq_S \beta$, if

$(*)$ there exist a left-c.e. approximation $\{a_s\}_{s\in\omega}$ of α, a left-c.e. approximation $\{b_s\}_{s\in\omega}$ of β, and a constant c such that $\alpha - a_s \leq c(\beta - b_s)$ for all s.

It is well-known that for left-c.e. reals α and β, Condition $(*)$ is equivalent to the following Condition $(*')$, and is equivalent to Condition $(*'')$ if β is not rational.

$(*')$ For any left-c.e. approximation $\{a'_s\}_{s\in\omega}$ of α, there exists a left-c.e. approximation $\{b'_s\}_{s\in\omega}$ of β and a constant c such that $\alpha - a'_s \leq c(\beta - b'_s)$ for all s.

$(*'')$ For any left-c.e. approximation $\{b''_s\}_{s\in\omega}$ of β, there exists a left-c.e. approximation $\{a''_s\}_{s\in\omega}$ of α and a constant c such that $\alpha - a''_s \leq c(\beta - b''_s)$ for all s.

Trivially, for left-c.e. reals α and β, each of the two latter conditions implies the first one. For a proof of the reverse implications, assume that condition $(*)$ holds for some constant c, and left-c.e. approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of reals α and β, respectively, where we can assume $a_0 = 0$. If α is rational, condition $(*')$ follows easily, details are left to the reader. Otherwise, given a left-c.e. approximation $\{a'_s\}_{s\in\omega}$ of α, for each s, let $f(s)$ be the largest index such that $a'_s \geq a_{f(s)}$. The function f is well-defined since $a_0 = 0$, and it is a computable order since by case assumption, all values a_s and a'_s differ from α. Condition $(*')$ then follows by letting $b'_s = b_{f(s)}$. Given a left-c.e. approximation $\{b''_s\}_{s\in\omega}$ of β, for each s, let $g(s)$ be the smallest index such that $b''_s \leq b_{g(s)}$. By assumption on β, all values b_s and b''_s differ from β, hence the function $g(s)$ is a well-defined computable order, and condition $(*'')$ follows by letting $a''_s = a_{g(s)}$.

These equivalences indicate that in a setting of left-c.e. reals, the notion of Solovay reducibility is robust and, intuitively speaking, $\alpha \leq_S \beta$ amounts to α being approximable by left-c.e. approximations at least as fast as β up to a constant factor.

1.2. Martin-Löf randomness

In algorithmic randomness, various concepts of random reals or, essentially equivalent, random infinite binary sequences are investigated. The concept that is most central and receives the most attention in the literature is Martin-Löf randomness, which can be defined in several equivalent ways; see the monograph by Downey and Hirschfeldt [3] for details. In what follows, the term random is used as a shorthand for Martin-Löf random.

1.3. Convergence rate of random left-c.e. reals

The degree structure induced by Solovay reducibility on the class of left-c.e. reals turned out to be rich and has been well studied [1, 2,4,5]. A fundamental result is the following theorem due to Solovay [1] and to Kučera and Slaman [6].

Theorem 1.2 (Solovay, Kučera and Slaman). *A left-c.e. real is Solovay complete if and only if it is Martin-Löf random.*

Here, a left-c.e. real is *Solovay complete* if every left-c.e. real is Solovay reducible to it. One way to interpret this theorem is that among left-c.e. reals exactly the random ones converge as slowly as possible.

The most well-known example of a random left-c.e. real is Chaitin's Ω [7], which is defined as the halting probability of a universal prefix-free Turing machine. In fact, it is known that every random left-c.e. real is the halting probability of some universal prefix-free machine [1,2,4,5]. As a consequence, all random left-c.e. reals share many properties with Ω, and accordingly such reals are called *Omega numbers*.

Since computable approximations to a random left-c.e. real converge so slowly, their speed of convergence can be set as a benchmark for the speed of convergence of other left-c.e. reals. This phenomenon was further explored by Barmpalias and Lewis-Pye [8] and by Miller [9].

Fix a random left-c.e. real Ω with a left-c.e. approximation $\{\Omega_s\}_{s\in\omega}$. Given a d.c.e. real α with a d.c.e. approximation $\{\alpha_s\}_{s\in\omega}$, let

$$\partial\alpha = \partial\{\alpha_s\} = \lim_{s\to\infty} \frac{\alpha - \alpha_s}{\Omega - \Omega_s}.$$

For this notion of $\partial\alpha$, Barmpalias and Lewis-Pye [8] showed the following result.

Theorem 1.3 (Barmpalias and Lewis-Pye). *For any random left-c.e. real α with a left-c.e. approximation $\{\alpha_s\}_{s\in\omega}$, the limit $\partial\alpha$ exists and is independent of the choice of the approximation $\{\alpha_s\}_{s\in\omega}$.*

Accordingly, $\partial\alpha$ is well-defined for any left-c.e. real α. Miller [9] observed that this result can be easily extended to all d.c.e. reals, in the following nicer form.

Theorem 1.4 (Barmpalias and Lewis-Pye, Miller). *For any d.c.e. real α, $\partial\alpha$ is well-defined and $\partial\alpha = 0$ if and only if α is not random.*

Theorem 1.4 implies that all the d.c.e. approximations to a random d.c.e. real converge at exactly the same speed.

1.4. Definition of speedability

Given the previous discussion of comparing the convergence rates of approximations of two different left-c.e. reals, it is natural to consider the convergence rates of different computable approximations of a single real, and the possibility of speeding up the convergence of a given computable approximation. This leads to the notion of *speedability* of a computable approximation.

Definition 1.5. Let $\{a_s\}_{s\in\omega}$ be a computable approximation that converges to a real α and let ρ be some rational in $[0,1)$. The approximation $\{a_s\}_{s\in\omega}$ is *ρ-speedable* if either α is rational or, if α is not rational, if there is a computable order f such that

$$\liminf_{s\to\infty}\left|\frac{\alpha - a_{f(s)}}{\alpha - a_s}\right| \leq \rho. \tag{1}$$

The approximation $\{a_s\}_{s\in\omega}$ is *speedable* if it is ρ-speedable for some $\rho \in [0,1)$.

Note that a computable approximation cannot be ρ-speedable for negative ρ and is always ρ-speedable for $\rho \geq 1$, hence, in connection with ρ-speedability, only values of ρ in the interval $[0,1)$ are relevant. For left-c.e. approximations, as long as ρ is nonzero, the actual choice of ρ within the latter interval does not matter by the following result of Merkle and Titov [10], Lemma 8.

Lemma 1.6 (Merkle and Titov). *If a left-c.e. approximation is ρ-speedable for some $\rho < 1$, then it is ρ-speedable for all $\rho \in (0,1)$.*

Remark 1.7. The proof in Merkle and Titov [10] only works for left-c.e. approximations, as it uses the monotonicity of the approximation in an essential way. However, by Theorem 2.1 an assertion similar to Lemma 1.6 holds for d.c.e. approximations: if a d.c.e. approximation is ρ-speedable for some $\rho < 1$, then there exists a 0-speedable d.c.e. approximation of the same real. The latter assertion remains true when both occurrences of d.c.e. are replaced by c.a.; we will, however, show the stronger result that all c.a. reals have a 0-speedable c.a. approximation.

Every c.a. real has a 0-speedable c.a. approximation, but if the real is random, it cannot have a speedable left-c.e. approximation. This suggests investigating the speedability of different types of computable approximations of the various types of c.a. reals.

Definition 1.8. Let α be a real, and ρ be some rational. Let X stand for one of the terms left-c.e., right-c.e., d.c.e., or c.a.

- The real α is X *(ρ-)speedable* if it has a (ρ-)speedable X approximation.
- The real α is *strongly* X *(ρ-)speedable* if it has an X approximation that is (ρ-)speedable via the function $s \mapsto s+1$.
- The real α is *fully* X *speedable* if it is X and all of its X approximations are speedable.
- The real α is *weakly* X *speedable* if there exist a rational $\rho \in (0,1)$ and two X approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of α such that

$$\liminf_{s\to\infty}\left|\frac{\alpha - b_s}{\alpha - a_s}\right| \leq \rho. \tag{2}$$

For all X $\in$ {left-c.e., right-c.e., d.c.e., c.a.}, the following hold.

$$\text{X 0-speedability} \Rightarrow \text{X speedability;} \tag{3}$$
$$\text{strong X speedability} \Rightarrow \text{X speedability;} \tag{4}$$
$$\text{full X speedability} \Rightarrow \text{X speedability;} \tag{5}$$
$$\text{X speedability} \Rightarrow \text{weak X speedability.} \tag{6}$$

The first three implications hold by definition. For a proof of (6), let α be a real that has an X approximation $\{a_s\}_{s\in\omega}$ that is speedable via some computable order f. Then $\{a_{f(s)}\}_{s\in\omega}$ is an X approximation of α by Remark 1.1, and the approximations $\{a_s\}_{s\in\omega}$ and $\{a_{f(s)}\}_{s\in\omega}$ witness that α is weakly X speedable.

1.5. Left-c.e. speedability

For left-c.e. in place of X, the implications in (4), (5), and (6) can be reversed.

Theorem 1.9 (Merkle and Titov). *For a left-c.e. real γ, the following are equivalent.*

(i) γ *is left-c.e. speedable.*
(ii) γ *is strongly left-c.e. ρ-speedable for all $\rho \in (0,1)$.*
(iii) γ *is fully left-c.e. speedable.*
(iv) γ *is weakly left-c.e. speedable.*

Remark 1.10. The equivalence of (i), (ii), and (iii) was shown by Merkle and Titov [10]. Concerning (iv), it follows as a special case of (6) that left-c.e. speedability implies weak left-c.e. speedability. For a proof of the reverse implication, we can assume that α is not rational. Suppose α satisfies the definition of weakly left-c.e. ρ-speedable via two left-c.e. approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$. We define a function $f : \mathbb{N} \to \mathbb{N}$ by letting $f(n)$ be the least number such that $a_{f(n)} \geq b_n$. It is easy to check that f is a well-defined computable order such that α is ρ-speedable via f.

Theorem 1.3 implies that left-c.e. random reals cannot be left-c.e. speedable. Following the latter observation, Merkle and Titov [10] asked whether this yields a characterization of the random left-c.e. reals.

Question 1. *Are all nonrandom left-c.e. reals left-c.e. speedable?*

This question was later answered in the negative by Hölzl and Janicki [11].

By Lemma 1.6, every speedable left-c.e. approximation is ρ-speedable for all $\rho > 0$. It is then natural to ask whether this inclusion can be extended to the case $\rho = 0$, i.e., whether (3) can be reversed for left-c.e. in place of X.

Question 2. *Are all left-c.e. speedable reals left-c.e. 0-speedable?*

This question was also answered in the negative by Hölzl, Janicki, Merkle, and Stephan [12].

1.6. Our results and outline of the paper

The main goal of this paper is to study the speedability notions in Definition 1.8 and their relation with randomness. Note that given a right-c.e. approximation $\{a_s\}_{s\in\omega}$ of a real α, we can consider the left-c.e. approximation $\{1 - a_s\}_{s\in\omega}$ and its limit $1 - \alpha$. Consequently, right-c.e. speedability is fully symmetric to left-c.e. speedability, and all results about left-c.e. speedability can be translated to right-c.e. speedability. In particular, Theorem 1.9 holds for right-c.e. in place of left-c.e. as well.

In Section 2, we will show that for d.c.e. in place of X, the implications (3), (4) and (6) can be reversed. Moreover, we will prove that a d.c.e. real is d.c.e. speedable if and only if it is nonrandom. Thus, Questions 1 and 2 both have positive answers in the d.c.e. case.

By the negative answer to Question 1 by Hölzl and Janicki [11], there is a nonrandom left-c.e. real that is not left-c.e. speedable. The latter real, on the one hand, it is not fully d.c.e. speedable, since it has a left-c.e. approximation which is not speedable and all left-c.e. approximations are also d.c.e. approximations; on the other hand, it is d.c.e. speedable by our result, since it is nonrandom. Thus, the reverse of the implication (5) does not hold for d.c.e. in place of X. However, as shown in Section 2.2, the reverse implication holds when restricting attention to d.c.e. reals that are neither left-c.e. nor right-c.e.

In Section 3, we study the speedability of c.a. reals. We first prove that every c.a. real is speedable. With a more involved argument, we then show that every c.a. real is strongly c.a. 0-speedable.

Though Question 2 has a negative answer, the following question is still of interest.

Question 3. *Which left-c.e. reals exactly are speedable?*

We would like to obtain a characterization of the left-c.e. speedable reals among the left-c.e. reals by some randomness notion. This would lead to a characterization of the fully d.c.e. speedable reals among the d.c.e. reals as well. In Section 4, as a step toward answering Question 3, we characterize the left-c.e. speedable reals as the reals that can be covered by suitably restricted Solovay tests.

1.7. Notation

For standard notation and definitions in computability and randomness, we refer to the monograph by Downey and Hirschfeldt [3]. In the remainder of this section, we recall some notation specific to this article and adopt some conventions.

The set of natural numbers, which includes 0, is usually denoted by $\mathbb{N}$, and is denoted by ω when used in subscripts. A real with binary expansion $0.b_0b_1b_2\ldots$ is identified with the infinite binary sequence $b_0{}^\frown b_1{}^\frown b_2\cdots$. For a dyadic rational, it is always assumed that its binary expansion ends with infinitely many zeros.

An approximation $\{a_s\}_{s\in\omega}$ of a real α is *two-sided* if there are infinitely many s such that $a_s < \alpha$ and infinitely many s such that $a_s > \alpha$. Such an approximation is *left-sided* if for all but finitely many s, $a_s \leq \alpha$, and it is *right-sided* if for all but finitely many s, $a_s \geq \alpha$.

Remark 1.11. For a real α that is neither left-c.e. nor right-c.e., all its computable approximations must be two-sided. For a proof by contradiction, assume that the real has a left-sided computable approximation $\{a_s\}_{s\in\omega}$, and fix an index t such that $a_s \leq \alpha$ for all $s \geq t$. Letting b_s be the maximum of $a_t, \ldots, a_{t+s}$ yields the left-c.e. approximation $\{b_s\}_{s\in\omega}$ of α, contrary to assumption. By a symmetric argument, α cannot have a right-sided computable approximation, either.

The term *string* always refers to a finite binary string. For a string σ, its *length* is denoted by $|\sigma|$. Given two strings σ and τ, we write $\sigma \prec \tau$, if σ is a proper prefix of τ. For a set A of strings, we say A is *prefix-free* if no string in A is a prefix of another string in A.

For a string σ, let $[\![\sigma]\!]$ denote the set of all infinite binary sequences that extend σ. For a set E of strings, we define $[\![E]\!] = \bigcup_{\sigma\in E}[\![\sigma]\!]$. Let μ denote the Lebesgue measure. For simplicity, we write $\mu([\![E]\!])$ as $\mu(E)$.

We identify a *Martin-Löf test* with a uniformly computable sequence $\{U_i\}_{i\in\omega}$ of prefix-free sets of strings such that $\mu(U_i) \leq 2^{-i}$ for all i. A real γ is *random* if there is no Martin-Löf test $\{U_i\}_{i\in\omega}$ such that $\gamma \in \bigcap_i [\![U_i]\!]$.

We also identify a *Solovay test* with a sequence $\{[\ell_i, r_i]\}_{i\in\omega}$ of nonempty intervals such that $\{\ell_i\}_{i\in\omega}$ and $\{r_i\}_{i\in\omega}$ are computable sequences of dyadic rationals in $[0,1]$ and the sum over the terms $r_i - \ell_i$ converges. It is known that a real γ is not random if and only if there is such a Solovay test such that $\gamma \in [\ell_i, r_i]$ for infinitely many i, and in this situation we say the real γ is covered by or fails the Solovay test.

2. D.c.e. speedability

2.1. D.c.e. speedability and randomness

Theorem 2.1. *For a d.c.e. real γ, the following are equivalent.*

(i) γ *is strongly d.c.e. 0-speedable.*
(ii) γ *is d.c.e. speedable.*
(iii) γ *is weakly d.c.e. speedable.*
(iv) γ *is nonrandom.*

Proof. The chain of implications (i) ⇒ (ii) ⇒ (iii) follows by (4) and (6), while the implications (iii) ⇒ (iv) and (iv) ⇒ (i) are precisely the statements of Lemmas 2.2 and 2.3, respectively. □

Lemma 2.2. *If γ is a random d.c.e. real, then γ is not weakly d.c.e. speedable.*

Proof. Let γ be a random d.c.e. real, and let $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ be any two d.c.e. approximations of γ. By Theorem 1.4, we have $\partial\{a_s\} = \partial\{b_s\} = \partial\gamma \neq 0$, i.e.,

$$\lim_{s\to\infty} \frac{\gamma - a_s}{\Omega - \Omega_s} = \lim_{s\to\infty} \frac{\gamma - b_s}{\Omega - \Omega_s} \neq 0.$$

Then

$$\lim_{s\to\infty} \frac{\gamma - b_s}{\gamma - a_s} = \lim_{s\to\infty} \frac{\frac{\gamma - b_s}{\Omega - \Omega_s}}{\frac{\gamma - a_s}{\Omega - \Omega_s}} = 1.$$

Thus, γ is not weakly d.c.e. speedable. □

Merkle and Titov [10] observed that it is immediate from Theorem 1.3 that a random left-c.e. real cannot be left-c.e. speedable. Moreover, they gave a direct combinatorial proof for the latter assertion. Similar to the left-c.e. case, here we also give a direct combinatorial proof of Lemma 2.2.

Proof. [Alternative proof of Lemma 2.2] For a proof by contradiction, assume that for some random real α, there are d.c.e. approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of α and $\rho \in (0,1)$ such that

$$|\alpha - b_s| \leq \rho|\alpha - a_s| \text{ for infinitely many } s. \tag{7}$$

By an argument of Rettinger [[3], Theorem 9.2.4], d.c.e. approximations of the random real α cannot be two-sided. Thus, the two approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ must be left-sided or right-sided. Actually, they must be either both left-sided or both right-sided; otherwise, the real α would be left-c.e. and right-c.e., hence computable. Without loss of generality, we can assume that both approximations are left-sided.

We define a function $f : \mathbb{N} \to \mathbb{N}$ by letting $f(0) = 1$ and for all $s \geq 0$,

$$f(s+1) = \min\{i > f(s) : a_i \geq b_{s+1}\}.$$

It is easy to check that f is well-defined and is a strictly increasing computable function where $f(s) \geq s+1$ and $a_{f(s)} \geq b_s$ for all s. Moreover, by (7), we have

$$\alpha - a_{f(s)} \leq \rho(\alpha - a_s) \text{ for infinitely many } s. \tag{8}$$

We partition the natural numbers into consecutive intervals $\{J_r\}_{r\in\omega}$, which are defined inductively. Let $J_0 = \{0\}$, and for $r \geq 0$ let

$$J_{r+1} = \{\max J_r + 1, \max J_r + 2, \dots, f(\max J_r)\}.$$

By construction and since the function f is strictly increasing, for all r and all s in J_r, the value $f(s)$ is in $J_r \cup J_{r+1}$. For all r let

$$c_r = \min\{a_s : s \in J_r\} \quad \text{and} \quad d_r = \max\{a_s : s \in J_r \cup J_{r+1}\}.$$

Now fix indices s and r such that s is in J_r, and note that $c_r \leq a_s$ and $d_r \geq a_{f(s)}$. Moreover, $a_j < \alpha$ holds for all j. Since every interval J_r is finite, by (8), for infinitely many r we have

$$\alpha - d_r \leq \rho(\alpha - c_r), \text{ i.e., } \alpha \leq c_r + \frac{1}{1-\rho}(d_r - c_r). \tag{9}$$

Let $I_r = [c_r, c_r + \frac{1}{1-\rho}(d_r - c_r)]$. Then $\alpha \in I_r$ for infinitely many r.

On the other hand, as $d_r \in \{a_s : s \in J_r \cup J_{r+1}\}$ and $c_r \in \{a_s : s \in J_r\}$, it follows by the triangle inequality that $d_r - c_r \leq \sum_{s \in J_r \cup J_{r+1}} |a_{s+1} - a_s|$. Then

$$\sum_{r \in \omega} (d_r - c_r) \leq \sum_{r \in \omega} \sum_{s \in J_r \cup J_{r+1}} |a_{s+1} - a_s| \leq 2 \sum_{s \in \omega} |a_{s+1} - a_s| < \infty.$$

Therefore, the sum of the lengths of the intervals I_r is finite. Thus $\{I_r\}_{r \in \omega}$ is a Solovay test that covers the random real α, a contradiction. □

Lemma 2.3. *If γ is a nonrandom d.c.e. real, then γ is strongly d.c.e. 0-speedable.*

Proof. We can assume that γ is not rational because, otherwise, $\{\gamma - 2^{-s^2}\}_{s \in \omega}$ is a strongly d.c.e. 0-speedable approximation to γ. Fix some d.c.e. approximation $\{r_j\}_{j \in \omega}$ of γ. By Remark 1.1, we can assume that the r_j are pairwise distinct dyadic rationals. Let τ_j be the unique string with no trailing zeros such that $r_j = 0.\tau_j$. As γ is not random, let $\{U_i\}_{i \in \omega}$ be a Martin-Löf test which covers γ, i.e., we have for all i

$$\mu(U_i) \leq 2^{-i} \quad \text{and} \quad \gamma \in [\![U_i]\!].$$

As $\{U_i\}_{i \in \omega}$ are uniformly c.e., we can assume that the sets U_i are pairwise disjoint. This can be achieved by modifying the uniform enumeration process as follows: whenever there is some string σ to be enumerated into U_j that has already appeared before in some other set U_i, instead of σ, enumerating into U_j all extensions of σ of a suitable length such that these strings have not appeared as far in the whole enumeration. Let $\sigma_0, \sigma_1, \ldots$ be an effective enumeration of all the strings in U_{2k} for all k.

We construct an approximation $\{c_s\}_{s \in \omega}$ of γ such that

$$\liminf_{s \to \infty} \left| \frac{\gamma - c_{2s+1}}{\gamma - c_{2s}} \right| = 0. \tag{10}$$

Construction: *At stage* 0*:* declare all strings σ_i except σ_0 to be unused, and let $i(0) = j(0) = 0$.
At stage $s > 0$*:* let $j(s)$ be the minimum index $j > j(s-1)$ such that for some $i \leq j$, the string σ_i is yet unused and a prefix of τ_j. (We show the existence of such indices i and j shortly in the verification part below.) Among all such indices i, let $i(s)$ be equal to the minimum one and declare $\sigma_{i(s)}$ to be used. We let $k(s)$ be the unique index such that $\sigma_{i(s)}$ is in $U_{2k(s)}$. Then we let

$$c_{2s} = r_{j(s)} + \delta_s \quad \text{and} \quad c_{2s+1} = r_{j(s)} \qquad \text{where } \delta_s = 2^{-|\sigma_{i(s)}| + k(s)}.$$

Verification: We call a string *true*, if it is in $\{\sigma_i\}_{i \in \omega}$ and is a prefix of γ. By choice of the U_i, each U_{2k} contains a true string. Then there are infinitely many true strings. As $\{r_t\}_{t \in \omega}$ converge to γ, thus each of the infinitely many true strings σ_r is a prefix of τ_t for almost all t. So in each stage there are indices j and i as required. Consequently, the sequences $\{i(s)\}_{s \in \omega}$ and $\{j(s)\}_{s \in \omega}$ are well-defined. Furthermore, both sequences are computable because the construction is effective. By definition, the sequence $\{j(s)\}_{s \in \omega}$ is strictly increasing. Since we never make a string σ_i unused again after it has been used, every string σ_i is used at most once.

For every true string σ_r, there must be a stage s such that σ_r is a prefix of τ_j for all $j \geq s$. Then if σ_r has not been used at stage s yet, by definition, σ_r must be used in at most r many more stages. Thus, every true string is used eventually.

We call a stage *true*, if a true string is used at this stage. Let s be a true stage. The strings $\tau_{j(s)}$ and γ share the prefix $\sigma_{i(s)}$, hence the reals $r_{j(s)} = c_{2s+1}$ and γ differ at most by $2^{-|\sigma_{i(s)}|}$. On the other hand, the values c_{2s} and $r_{j(s)}$ differ by $\delta_s = 2^{-|\sigma_{i(s)}| + k(s)}$, hence c_{2s} and γ differ by at least $(2^{k(s)} - 1)2^{-|\sigma_{i(s)}|}$. Thus,

$$\left| \frac{\gamma - c_{2s+1}}{\gamma - c_{2s}} \right| \leq \frac{1}{2^{k(s)} - 1} \text{ for every true stage } s. \tag{11}$$

Since each U_{2k} is prefix-free, it contains only one true string. On the other hand, in every stage the string used is distinct. Then $k(s)$ are different for different true stages s. Thus, (11) implies (10), as there are infinitely many true stages.

To complete the proof, it remains to show that the sequence $\{c_i\}_{i \in \omega}$ is a d.c.e. approximation of γ. It suffices to show that the sum over the terms $|c_{i+1} - c_i|$ is finite. Note that the latter implies, in particular, that the sequence $\{c_i\}_{s \in \omega}$ converges, hence converges to the same limit γ as its subsequence $\{r_{j(s)}\}_{s \in \omega}$. By definition, we have

$$|c_{2s+1} - c_{2s}| \leq \delta_s \quad \text{and} \quad |c_{2s+2} - c_{2s+1}| \leq |r_{j(s+1)} - r_{j(s)}| + \delta_{s+1}.$$

By the triangle inequality and because $\{r_{j(s)}\}_{s \in \omega}$ is an increasing subsequence of the d.c.e. approximation $\{r_j\}_{j \in \omega}$, the sum over the terms $|r_{j(s+1)} - r_{j(s)}|$ is finite. The sum over the δ_s is also finite since each σ_i is used at most once. We have

$$\sum_{s \in \omega} \delta_s = \sum_{s \in \omega} 2^{-|\sigma_{i(s)}| + k(s)} \leq \sum_{k \in \omega} \sum_{\sigma \in U_{2k}} 2^{-|\sigma| + k} \leq \sum_{k \in \omega} 2^k \sum_{\sigma \in U_{2k}} 2^{-|\sigma|} \leq \sum_{k \in \omega} 2^{-k} = 2.$$

□

2.2. Full d.c.e. speedability

Lemma 2.4. *Every two-sided computable approximation is speedable.*

For the proof, we need the following technical fact.

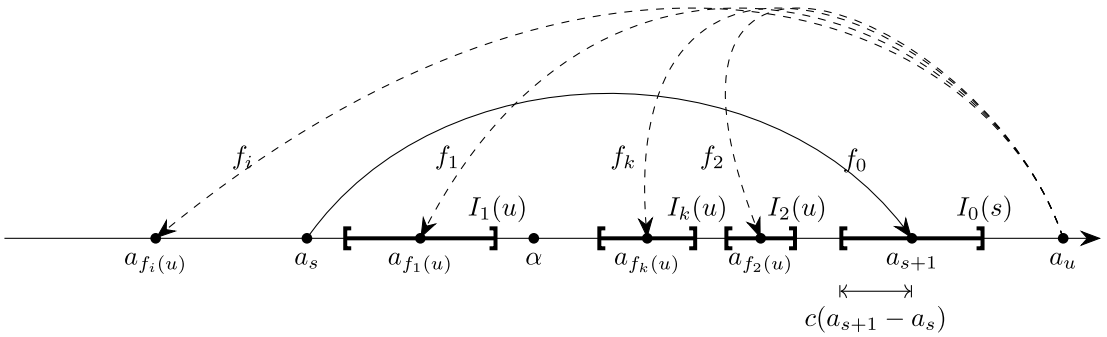

Fig. 1. $a_{f_1(u)}, a_{f_2(u)}, \ldots, a_{f_k(u)}$ jump over the interval $[a_{s+1} - c(a_{s+1} - a_s), a_{s+1}]$, thus there exists $i \in \{1, \ldots, k\}$ such that $a_{f_i(u)} < a_s$.

Proposition 2.5. *Given an interval $[a, b]$ and an array of intervals $I_0 = [z_0 - d_0, z_0 + d_0]$, …, $I_m = [z_m - d_m, z_m + d_m]$, if $d_i \geq (b-a)/m$ for every $0 \leq i \leq m$ and $z_j \notin I_i$ for all $0 \leq i < j \leq m$, then there exists $i \in \{0, \ldots, m\}$ such that $z_i \notin [a, b]$.*

Proof. For all $0 \leq i < j \leq m$, as $z_j \notin I_i$, we have $|z_j - z_i| > d_i \geq (b-a)/m$. Rearrange $z_0, \ldots, z_m$ in order of magnitude such that $z_{\pi(0)} < \cdots < z_{\pi(m)}$. Then $z_{\pi(i)} - z_{\pi(i-1)} > (b-a)/m$ for all $0 < i \leq m$. Thus,

$$z_{\pi(m)} - z_{\pi(0)} = \sum_{i=1}^{m} (z_{\pi(i)} - z_{\pi(i-1)}) > m \cdot \frac{b-a}{m} = b - a.$$

Therefore, the interval $[a, b]$ cannot cover both $z_{\pi(m)}$ and $z_{\pi(0)}$. □

Proof. [Proof of Lemma 2.4] Given $\{a_s\}_{s\in\omega}$ as in the hypothesis and $\rho \in (0, 1)$, for a contradiction suppose that $\{a_s\}_{s\in\omega}$ is not ρ-speedable.

Let $c = \rho/(1+\rho)$ and k be an integer such that $k \geq 1 + c^{-2}$. For every $i \in \{0, \ldots, 2k\}$, we inductively define a computable order f_i and a constant n_i such that

$$n_i \geq n_{i-1} \text{ and } \alpha \notin I_i(n) \text{ for all } n \geq n_i, \tag{12}$$

where

$$n_{-1} = 0 \text{ and } I_i(n) = \left[a_{f_i(n)} - c|a_{f_i(n)} - a_n|,\ a_{f_i(n)} + c|a_{f_i(n)} - a_n|\right].$$

First, let

$$f_0(n) = n + 1.$$

As $\{a_s\}_{s\in\omega}$ is not ρ-speedable, there exists an index s such that, for every $n \geq s$, it holds that $\left|\frac{\alpha - a_{f_0(n)}}{\alpha - a_n}\right| > \rho$. It implies that $\alpha \notin I_0(n)$. We define n_0 to be the least such index s. Thus, (12) is satisfied for $i = 0$.

Given $1 \leq i \leq 2k$, suppose f_j and n_j are defined and satisfy (12) for all $j \in \{0, \ldots, i-1\}$. Then for all $n \geq n_{i-1}$ we have $\alpha \notin \bigcup_{j=0}^{i-1} \bigcup_{\ell=n_j}^{n} I_j(\ell)$. Thus,

$$f_i(n) = \begin{cases} n+1 & \text{if } n < n_{i-1}, \\ \min\{m > n : a_m \notin \bigcup_{j=0}^{i-1} \bigcup_{\ell=n_j}^{n} I_j(\ell)\} & \text{otherwise.} \end{cases} \tag{13}$$

is well-defined. It is easy to check that f_i is a strictly increasing function. Now that f_i is not a ρ-speedup function for $\{a_s\}_{s\in\omega}$, there exists an index $s \geq n_{i-1}$ such that, for every $n \geq s$, it holds that $\left|\frac{\alpha - a_{f_i(n)}}{\alpha - a_n}\right| > \rho$, which implies that $\alpha \notin I_i(n)$. We define n_i to be the least such index s. Then (12) is satisfied for i. This completes the inductive definition of f_i and n_i for all $i \in \{0, \ldots, 2k\}$.

As $\{a_s\}_{s\in\omega}$ is a two-sided approximation to α, let $s \geq n_{2k}$ be an index such that $a_s < \alpha < a_{s+1}$.

Now we derive a contradiction as promised. The proof idea is as follows. We first find the index $u \geq s+1$ such that a_u is the last one to the right of or equal to a_{s+1}. Then all of $a_{f_1(u)}, a_{f_2(u)}, \ldots, a_{f_k(u)}$ are to the left of a_{s+1}. Moreover, they must jump over the interval $[a_{s+1} - c(a_{s+1} - a_s), a_{s+1}]$. Then their distances from a_u are bounded below. Thus, by our arrangement of the parameters, we argue that at least one of them must be to the left of a_s. This is illustrated in Fig. 1.

Fixing one such point $a_{f_i(u)}$, we find the index $v \geq f_i(u)$ such that a_v is the last one to the left of or equal to $a_{f_i(u)}$. In a symmetrical way as before, we find one point $a_{f_{i+j}(v)}$ to be to the right of a_u. However, this contradicts the fact that a_u is the last one to the right of or equal to a_{s+1}. The whole argument is illustrated in Fig. 2.

Formally, let $u \geq s+1$ be the index such that

$$a_u \geq a_{s+1} \text{ and } \forall t > u (a_t < a_{s+1}). \tag{14}$$

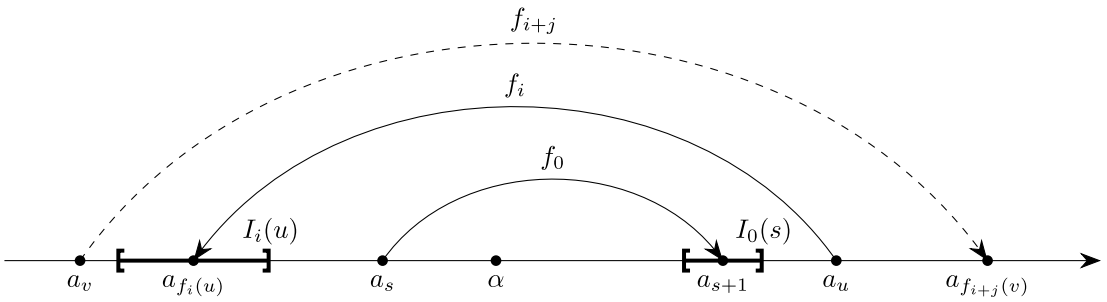

Fig. 2. We first find a_u to be the last one to the right of or equal to a_{s+1}. Then find $a_{f_i(u)}$ to the left of a_s, and then a_v the last one to the left of or equal to $a_{f_i(u)}$, and then $a_{f_{i+j}(v)}$ to the right of a_u. This contradicts the fact that a_u is the last one to the right of or equal to a_{s+1}.

The existence of such u is guaranteed by the fact that $\{a_s\}_{s\in\omega}$ converges to α and $\alpha < a_{s+1}$. For each $i \in \{1,\dots,k\}$, by definition of f_i and $s \geq n_{2k} \geq n_0$ we have

$$a_{f_i(u)} \notin I_0(s) = \big[a_{s+1} - c(a_{s+1} - a_s),\, a_{s+1} + c(a_{s+1} - a_s)\big].$$

On the other hand, as $f_i(u) > u$, by (14) we have $a_{f_i(u)} < a_{s+1}$. Thus, $a_{f_i(u)} < a_{s+1} - c(a_{s+1} - a_s)$. Then

$$c|a_{f_i(u)} - a_u| \geq c(a_{s+1} - a_{f_i(u)}) > c^2(a_{s+1} - a_s) \geq (a_{s+1} - a_s)/(k-1).$$

Moreover, by definition, $a_{f_j(u)} \notin I_i(u)$ for all $0 \leq i < j \leq k$. Thus, the interval $[a_s, a_{s+1}]$ and the array of intervals $I_1(u)$, $I_2(u)$, …, $I_k(u)$ satisfy the condition of Proposition 2.5. Therefore there exists $i \in \{1,\dots,k\}$ such that $a_{f_i(u)} \notin [a_s, a_{s+1}]$, i.e. , $a_{f_i(u)} < a_s$.

Fix one such i. Similar as before, let $v \geq f_i(u)$ be the index such that

$$a_v \leq a_{f_i(u)} \text{ and } \forall t > v(a_t > a_{f_i(u)}). \tag{15}$$

The existence of such v is guaranteed by the fact that $\{a_s\}_{s\in\omega}$ converges to α and $\alpha > a_s > a_{f_i(u)}$. For each $j \in \{1,\dots,k\}$, by definition and $u \geq s \geq n_{2k} \geq n_i$ we have

$$a_{f_{i+j}(v)} \notin I_i(u) = \big[a_{f_i(u)} - c(a_u - a_{f_i(u)}),\, a_{f_i(u)} + c(a_u - a_{f_i(u)})\big].$$

On the other hand, as $f_{i+j}(v) > v$, by (15), $a_{f_{i+j}(v)} > a_{f_i(u)}$. Thus, $a_{f_{i+j}(v)} > a_{f_i(u)} + c(a_{f_i(u)} - a_u)$. Then

$$c|a_{f_{i+j}(v)} - a_v| \geq c(a_{f_{i+j}(v)} - a_{f_i(u)}) > c^2(a_{f_i(u)} - a_u) \geq (a_{f_i(u)} - a_u)/(k-1).$$

Moreover, by definition, $a_{f_{i+j}(v)} \notin I_{i+\ell}(v)$ for all $0 \leq \ell < j \leq k$. Thus, the interval $[a_{f_i(u)}, a_u]$ and the array of intervals $I_{i+1}(v)$, $I_{i+2}(v)$, …, $I_{i+k}(v)$ satisfy the condition of Proposition 2.5. Therefore, there exists $j \in \{1,\dots,k\}$ such that $a_{f_{i+j}(v)} \notin [a_{f_i(u)}, a_u]$. However, by (14) and (15), for all $t > v$, $a_t \in (a_{f_i(u)}, a_{s+1}) \subset [a_{f_i(u)}, a_u]$. As $f_{i+j}(v) > v$, this is a contradiction. □

The following theorem is immediate by Remark 1.11 and Lemma 2.4.

Corollary 2.6. *If a d.c.e. real is neither left-c.e. nor right-c.e., then it is fully d.c.e. speedable.*

3. C.a. speedability

Similar to the d.c.e. case, by Remark 1.11 and Lemma 2.4, we also have the following result about fully c.a. speedability.

Theorem 3.1. *If a c.a. real is neither left-c.e. nor right-c.e., then it is fully c.a. speedable.*

As for c.a. speedability, with the following proposition, Lemma 2.4 actually implies that all c.a. reals are c.a. speedable.

Proposition 3.2. *Every c.a. real has a two-sided computable approximation.*

Proof. This is obvious if α is rational, so from now on, we can assume otherwise. Fix some computable approximation $\{a_s\}_{s\in\omega}$ of α where, by Remark 1.1, we can assume that the a_s are pairwise distinct dyadic rationals in the interval $(0,1)$. For each $s > 0$, let $h(s)$ be the position of the most significant bit at which a_s and a_{s-1} differ, where, as usual, more significant means closer to the decimal point and counting of positions starts with 1. Let

$$a_s^- = \max\{a_s - 2^{-h(s)}, 0\} \quad \text{and} \quad a_s^+ = \min\{a_s + 2^{-h(s)}, 1\}.$$

As $\{a_s\}_{s\in\omega}$ converges to the irrational real α and the a_s are pairwise distinct, the values $h(s)$ tend to infinity. Then there are infinitely many stages s such that $h(t) > h(s)$ for all $t > s$. To see this, assume otherwise, then there is s_0 such that for all $s \geq s_0$, there exists $t > s$ with $h(t) \leq h(s)$. Then $\liminf_{s\to\infty} h(s) \leq h(s_0)$, contradicts to the fact that $\lim_{s\to\infty} h(s) = \infty$. Now for each such stage s, a_s and α agree on the bits up to position $h(s)$. Clearly, $a_s^- < \alpha < a_s^+$. In summary, the sequence $\{a_1^-, a_1^+, a_2^-, a_2^+, \dots\}$ is a two-sided computable approximation of α. □

Corollary 3.3. *Every c.a. real is c.a. speedable.*

With a more involved argument, we show the following stronger result.

Theorem 3.4. *Every c.a. real is strongly c.a. 0-speedable.*

Proof. Let α be a c.a. real. Fix some computable approximation $\{a_s\}_{s\in\omega}$ of α. By Remark 1.1, we can assume that the a_s are pairwise distinct dyadic rationals. Observe that since the values a_i converge to α, for every pair of fixed indices $s > 0$ and $g(s) > s$ it holds that

$$\left|\frac{\alpha - a_{g(s)}}{\alpha - a_s}\right| < \frac{1}{s} \quad \text{if and only if} \quad \left|\frac{a_i - a_{g(s)}}{a_i - a_s}\right| < \frac{1}{s} \quad \text{for all sufficiently large } i. \tag{16}$$

We construct in stages a computable approximation $\{b_s\}_{s\in\omega}$ of α that is c.a. 0-speedable via the function $s \mapsto s+1$. At each stage $i > 0$, we maintain a function $g : \mathbb{N} \to \mathbb{N}$ with $g(s) \geq s$ for all s. The construction resembles a finite-injury priority argument. We say that index $s < i$ *requires attention* at stage i if s and the current value of $g(s)$ do not satisfy the inequality on the right-hand side of (16).

Construction: *At stage* 0*:* we initialize the function g by letting $g(s) = s$ for all s.
At stage $i > 0$*:* let $s \leq i$ be the minimum such that s requires attention. We increment the value $g(s)$ by one, and let $b_{2i} = a_s$ and $b_{2i+1} = a_{g(s)}$. We say the index s *receives attention* at this stage.

Verification: Note that at stage $i > 0$ an index $s < i$ that requires attention always exists: at the beginning of stage i there is some $0 < s \leq i$ where $g(s) = s$ since in the previous stages at most $i-1$ values of g were incremented.

Now we show that each index s receives attention at most finitely often. Indeed, since the a_i converge to α, for every s there is a natural number $m_s > s$ such that for $\delta = |\alpha - a_s| > 0$, and for all $i, t \geq m_s$,

$$|a_i - a_s| > \frac{\delta}{2} \quad \& \quad |a_i - a_t| < \frac{\delta}{2s}, \quad \text{which implies} \quad \left|\frac{a_i - a_t}{a_i - a_s}\right| < \frac{1}{s}.$$

Thus s will receive attention at most $m_s - s$ times, after which $g(s) \geq m_s$ holds.

Fix some index t, and let i be a stage where $b_{2i} = a_t$ or $b_{2i+1} = a_t$. Then either t or an index $s < t$ where $t = g(s)$ receives attention at stage i, which can happen at most finitely often. Thus, each value a_t occurs at most finitely often in the sequence $\{b_s\}_{s\in\omega}$. Then the sequence $\{b_s\}_{s\in\omega}$ converges to α and is a computable approximation of α by construction.

Finally, we show that the approximation $\{b_s\}_{s\in\omega}$ is c.a. 0-speedable via the function $s \mapsto s+1$. For each s, let t_s be the stage after which index s never receives attention again. Then $t_s \geq s$, and s and the final value of $g(s)$ do satisfy the inequality on the right-hand side of (16) for all $i > t_s$, which then implies the inequality on the left-hand side of (16). In other words, for each s, there exists $t_s \geq s$ such that

$$\left|\frac{\alpha - b_{2t_s+1}}{\alpha - b_{2t_s}}\right| < \frac{1}{s}, \quad \text{which then implies} \quad \liminf_{s\to\infty} \left|\frac{\alpha - b_{s+1}}{\alpha - b_s}\right| = 0.$$

□

4. Left-c.e. speedability and randomness

As discussed in Section 1.5, the left-c.e. speedable reals form a proper subclass of the nonrandom left-c.e. reals [10,11]. As a step towards a characterization of the left-c.e. speedable reals via notions of randomness, we demonstrate in this section that a real is left-c.e. speedable if and only if it is covered by a restricted type of Solovay test. Before that, we first discuss the connections between effective approximations and Solovay tests in general. This relates to the observation of Rettinger [[3], Theorem 9.2.4] that given a two-sided d.c.e. approximation $\{a_s\}_{s\in\omega}$ of some real, this real is covered by the Solovay test that is formed by the intervals $[\min\{a_s, a_{s+1}\}, \max\{a_s, a_{s+1}\}]$.

Definition 4.1. Let $\{I_i\}_{i\in\omega}$ be a Solovay test where $I_i = [\ell_i, r_i]$. Note that by our convention, $0 \leq \ell_i \leq r_i \leq 1$ for all i. We refer to $\ell_0, r_0, \ell_1, r_1, \ldots$ as the *sequence of endpoints* of this Solovay test, and its *sequence of left endpoints* and *sequence of right endpoints* are defined accordingly. The Solovay test then

- is *converging* if its sequence of endpoints converges,
- is *d.c.e.* if its sequence of endpoints has bounded variation,
- is *left-c.e.* if its sequence of left endpoints is nondecreasing,
- *has bounded increments* if for some rational constant $d > 0$ it holds for all i that

$$\ell_{i+1} \geq \ell_i + d(r_i - \ell_i). \tag{17}$$

By definition, it is easy to check that Solovay tests with bounded increments are left-c.e., and d.c.e. Solovay tests are converging. With the following observation, left-c.e. Solovay tests are d.c.e.

$$\sum_i (|r_i - \ell_i| + |\ell_{i+1} - r_i|) \leq \sum_i (|r_i - \ell_i| + |\ell_{i+1} - \ell_i| + |\ell_i - r_i|) = 2\sum_i |r_i - \ell_i| + \sum_i |\ell_{i+1} - \ell_i|.$$

By *limit* of a converging Solovay test, we refer to the limit of its sequence of endpoints.

Remark 4.2. A converging Solovay test cannot cover any real other than its limit. For a proof, assume that some real γ is covered by a converging Solovay test $\{I_i\}_{i\in\omega}$. Then there are infinitely many intervals I_i that contain γ, and since the interval lengths tend to 0, the left endpoints of these intervals converge to γ. So the sequence of endpoints converges to γ since it converges and has a subsequence that converges to γ. Note that a converging Solovay test may fail to cover its limit, and then does not cover any real at all. For example, for a strictly increasing left-c.e. approximation $\{a_s\}_{s\in\omega}$, the Solovay test $\{[a_s, a_{s+1}]\}_{s\in\omega}$ is converging but does not cover any real.

Remark 4.3. Consider a converging Solovay test that covers a real γ. Then the sequence of endpoints of this test is computable and converges to γ, hence is a c.a. approximation of γ. Similarly, if the Solovay test is d.c.e., then the sequence of endpoints is a d.c.e. approximation of γ, and if the Solovay test is left-c.e., then its sequence of left endpoints is a left-c.e. approximation of γ.

Proposition 4.4.

(i) A real is nonrandom and c.a. if and only if it is covered by a converging Solovay test.
(ii) A real is nonrandom and d.c.e. if and only if it is covered by a d.c.e. Solovay test.
(iii) A real is nonrandom and left-c.e. if and only if it is covered by a left-c.e. Solovay test.

Proof. For all three equivalences, the right-to-left direction is immediate by Remark 4.3 and since being nonrandom is equivalent to being covered by a Solovay test.

For a proof of the left-to-right direction of (i), let the real α be nonrandom and c.a. Let $\{I_i\}_{i\in\omega}$ be a Solovay test that covers α and let $\{a_i\}_{i\in\omega}$ be a c.a. approximation of α. We can assume that α is not rational, since otherwise there is nothing to prove. We can assume further that the intervals I_i are pairwise distinct. We define a sequence of intervals by a construction in stages $s = 0, 1, \ldots$. Initially, call all intervals I_i unused. At stage s, if there is some index $i \le s$ such that a_s is in the currently unused interval I_i, append the interval with the least such index to the constructed sequence and declare this interval to be used. The constructed sequence of intervals is indeed a Solovay test since the construction is effective, the sequence contains only intervals from the given Solovay test, and each such interval is appended at most once. The constructed Solovay test covers α because every interval I_i that contains α also contains almost all a_s, thus it follows by an easy inductive argument that every such intervals will eventually be appended to the constructed sequence. Finally, the constructed Solovay test is converging, since the a_s converge and the length of the intervals that are appended during some stage s tends to 0 when s tends to infinity.

In the proof of the left-to-right direction of (ii), we apply the same construction as above. However, now we choose $\{a_i\}_{i\in\omega}$ as a d.c.e. approximation of α. Let the constructed sequence of intervals be $\{J_i\}_{i\in\omega}$, where $J_i = [\ell_i, r_i]$. We have already seen that $\{J_i\}_{i\in\omega}$ is a Solovay test that covers α, so it suffices to show that this test is actually d.c.e. Let s_i be the stage at which J_i is appended, hence a_{s_i} is a member of J_i for all i. Consequently, we have

$$|\ell_{i+1} - r_i| = |\min J_{i+1} - \max J_i| \le |a_{s_{i+1}} - a_{s_i}| + |J_i| + |J_{i+1}|,$$

which implies

$$\sum_{i\in\omega} \underbrace{|r_i - \ell_i|}_{=|J_i|} + |\ell_{i+1} - r_i| \le 3\sum_{i\in\omega} |J_i| + \sum_{i\in\omega} |a_{s_{i+1}} - a_{s_i}| \le 3\sum_{i\in\omega} |J_i| + \sum_{i\in\omega} |a_{i+1} - a_i|. \tag{18}$$

Note that the last inequality in (18) holds by the triangle inequality. Since $\{J_i\}_{i\in\omega}$ is a Solovay test, and the sequence $\{a_i\}_{i\in\omega}$ has bounded variation, the upper bound in (18) is finite. As a consequence, the sequence of endpoints $r_0, \ell_0, r_1, \ell_1, \ldots$ of the Solovay test $\{J_i\}_{i\in\omega}$ has finite variation, hence the test is d.c.e.

In the proof of the left-to-right direction of (iii), we apply a variant of the same construction, where now we choose $\{a_i\}_{i\in\omega}$ as a left-c.e. approximation of α. The only difference is that whenever in the previous construction during some stage s some interval I would be appended, we append instead the interval $I \cap [a_s, 1]$. Observe that the latter interval contains α if and only if I contains α since we have $a_s < \alpha < 1$. Consequently, the construction yields again a Solovay test that covers α, which is obviously left-c.e. □

Proposition 4.5. *A real is left-c.e. speedable if and only if it is covered by a Solovay test that has bounded increments.*

Proof. Let α be a left-c.e. speedable real. Then α is strongly left-c.e. $\frac{1}{3}$-speedable by Theorem 1.9. Let $\{a_s\}_{s\in\omega}$ be a left-c.e. approximation of α which is $\frac{1}{3}$-speedable via the function $s \mapsto s+1$. Then for infinitely many s, $\alpha - a_{s+1} \le \frac{1}{3}(\alpha - a_s)$. If we let $I_s = [a_s, a_s + 2(a_{s+1} - a_s)]$, then $\alpha \in I_s$ for each such s, i.e., α is covered by the test $\{I_s\}_{s\in\omega}$. This test has finite measure $2(\alpha - a_0)$, hence is a Solovay test, and by construction has bounded increments with constant $d = \frac{1}{2}$.

For the converse, let α be covered by some Solovay test $\{[\ell_i, r_i]\}_{i\in\omega}$ that has bounded increments for some constant $d > 0$. Then the nondecreasing sequence $\ell_0, \ell_1, \ldots$ converges to α, hence is a left-c.e. approximation to α. Let i be any of the infinitely many indices such that $\alpha \in [\ell_i, r_i]$. We have $\ell_i \le \ell_{i+1} \le \alpha \le r_i$ and $\ell_{i+1} - \ell_i \ge d(r_i - \ell_i) \ge d(\alpha - \ell_i)$, which implies $d \le 1$ and

$$\frac{\alpha - \ell_{i+1}}{\alpha - \ell_i} \le \frac{\alpha - (\ell_i + d(r_i - \ell_i))}{\alpha - \ell_i} = 1 - d.$$

Consequently, $\{\ell_i\}_{i\in\omega}$ is $(1-d)$-speedable, hence α is left-c.e. speedable. □

CRediT authorship contribution statement

George Barmpalias: Investigation, Writing – review & editing; **Nan Fang:** Investigation, Writing – original draft, Writing – review & editing; **Wolfgang Merkle:** Investigation, Writing – review & editing; **Ivan Titov:** Investigation, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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