

Solovay Functions and the No-gap Phenomena



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infinitely often upper bound of K and C

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- $\forall x C(x) \leq^+ |x|$.
- $\exists^\infty x C(x) \geq^+ |x|$.

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We say that function $|x|$ is an *infinitely often tight upper bound* of C, up to a constant. How about prefix-free Kolmogorov complexity function K?

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We say that function $|x|$ is an *infinitely often tight upper bound* of C, up to a constant. How about prefix-free Kolmogorov complexity function K?

Definition

A function g is a *Solovay function* if g is computable and it holds that

- 1 $\forall x [K(x) \leq^+ g(x)]$
- 2 $\exists^\infty x [K(x) \geq^+ g(x)]$

A function g is a *weak Solovay function* if g is right-c.e. and satisfies both 1 and 2.

An equivalent characterization for Solovay functions

Theorem

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a right-c.e. function. Then f is an upper bound of $\mathcal{K}$ iff $\sum_n 2^{-f(n)}$ is finite.

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Theorem (Bienvenu and Downey, 2009)

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a right-c.e. function. Then f is a weak Solovay function $\Leftrightarrow \sum_n 2^{-f(n)}$ is finite and is a Martin-Löf random real.

K-triviality and Solovay functions

Definition

A sequence A is *K-trivial* if $\forall n \ K(A \upharpoonright n) \leq^+ K(n)$.

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Actually, we can replace $K(n)$ in the definition by any weak Solovay function.

Theorem (Bienvenu, Merkle and Nies, 2011)

If g is a (weak) Solovay function, then (*) a sequence A is *K-trivial* iff $\forall n \, K(A \upharpoonright n) \leq^+ g(n)$.

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Theorem (Bienvenu, Merkle and Nies, 2011)

If g is a (weak) Solovay function, then () a sequence A is K-trivial iff $\forall n \, K(A \upharpoonright n) \leq^+ g(n)$.*

And (*) turns out to be a characterization of Solovay function among all right-c.e. functions.

Theorem (Bienvenu, Downey, Nies and Merkle, 2015)

If g is a computable (right-c.e.) function such that for any sequence A , A is K-trivial iff $\forall n \, K(A \upharpoonright n) \leq^+ g(n)$, then g is a (weak) Solovay function.

Gács-Miller-Yu theorem and Solovay functions

Theorem (Gács-Miller-Yu)

*A sequence A is Martin-Löf random iff for all $n \in \omega$,
 $C(A \upharpoonright n) \geq^+ n - K(n)$.*

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Weak lowness for K and Solovay functions

Definition

- A sequence A is *weakly low for K* if $\exists^\infty n K^A(n) \geq K(n)$;
- A sequence A is *low for Ω* if Ω is Martin-Löf -random relative to A .

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Miller first showed that these two lowness are equivalent, while Bienvenu noticed a simple proof using Solovay function:

- Function K is right-c.e., it is also right-c.e. relative to A .
- And K is also an upper bound for K^A up to an additive constant.
- By definition, A is weakly low for K iff K is a weak Solovay function relative to A .
- Relativizing the equivalent characterization of Solovay function, K is a weak Solovay function relative to A iff $\Omega_K = \sum_n 2^{-K(n)}$ is Martin-Löf random relative to A .
- So A is weakly low for K iff A is low for Ω .

Weak lowness for K and Solovay functions

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- One direction is trivial.
- A is weakly low for K, then it is low for Ω .
- $\Omega_g = \sum_n 2^{-g(n)}$ is 1-random and left-c.e., then by Kučera-Slaman Theorem, it is Ω -like.
- Then Ω_g is 1-random relative to A .
- By relativization, $\exists^\infty n K^A(n) \geq^+ g(n)$.

Weak lowness for K and Solovay functions

Theorem

Let g be a right-c.e. function such that for any sequence A , A is weakly low for K iff $\exists^\infty n K^A(n) \geq^+ g(n)$, then g is a weak Solovay function.

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Theorem

Let g be a right-c.e. function such that for any sequence A , A is weakly low for K iff $\exists^\infty n K^A(n) \geq^+ g(n)$, then g is a weak Solovay function.

- For all sequence A , $\forall n K^A(n) \leq^+ K(n)$.
- If for some sequence A , $\exists^\infty n K^A(n) \geq^+ g(n)$, then $\exists^\infty n K(n) \geq^+ g(n)$.
- If g is not an upper bound of K , then $\sum_n 2^{-g(n)} = \infty$.
- For all A , $\sum_n 2^{-K^A(n)} < \infty$, $\exists^\infty n K^A(n) \geq^+ g(n)$.

2-randomness and Solovay functions

Theorem (Miller)

A set A is 2-random iff $\exists^\infty n K(A \upharpoonright n) \geq^+ K(n) + n$.

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If g is a weak Solovay function, then a sequence A is 2-random iff $\exists^\infty n K(A \upharpoonright n) \geq^+ n + f(n)$.

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If g is a weak Solovay function, then a sequence A is 2-random iff $\exists^\infty n K(A \upharpoonright n) \geq^+ n + f(n)$.

- A is 2-random iff A is 1-random and low for Ω .
- A is 1-random, by Ample Excess Lemma,
 $\forall n K^A(n) \leq^+ K(A \upharpoonright n) - n$.
- A is low for Ω , by previous result, $\exists^\infty n K^A(n) \geq^+ g(n)$.
- Thus, $\exists^\infty n K(A \upharpoonright n) \geq^+ n + g(n)$.

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If f is a right-c.e. function, and for any sequence A , A is 2-random iff $\exists^\infty n \ K(A \upharpoonright n) \geq^+ n + f(n)$, then f is a weak Solovay function.

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If f is a right-c.e. function, and for any sequence A , A is 2-random iff $\exists^\infty n K(A \upharpoonright n) \geq^+ n + f(n)$, then f is a weak Solovay function.

- For all sequence A , $\forall n K(A \upharpoonright n) \leq^+ n + K(n)$.
- If for some sequence A , $\exists^\infty n K(A \upharpoonright n) \geq^+ n + f(n)$, then $\exists^\infty n K(n) \geq^+ f(n)$.
- If g is not an upper bound of K , then for all A , $\exists^\infty n K^A(n) \geq^+ f(n)$.
- By Ample Excess Lemma, then all 1-random sequences A , $\exists^\infty n K(A \upharpoonright n) \geq^+ n + f(n)$.

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It seems very promising that in the definition the function $K(n)$ can be replaced by arbitrary Solovay function, but we will see that it is false.

Theorem

There is a Solovay function f that for some sequence A there are infinitely many point n such that $K(A \upharpoonright n) \leq^+ f(n)$ but A is not infinitely often K-trivial

Infinitely often K-triviality and Solovay functions

Proof.

- Suppose f is a Solovay function, define f_1 and f_2 as follows:

$$f_1(x) = \begin{cases} f(x) & \text{if } x \text{ is odd} \\ 2x & \text{if } x \text{ is even} \end{cases} \quad f_2(x) = \begin{cases} 2x & \text{if } x \text{ is odd} \\ f(x) & \text{if } x \text{ is even} \end{cases}$$

- $\forall x K(x) \leq^+ 2|x| \leq 2x$, and $\forall x K(x) \leq^+ f(x)$, then $\forall x K(x) \leq^+ f_1(x)$ and $K(x) \leq^+ f_2(x)$.
- As there are infinitely many x such that $K(x) \geq^+ f(x)$. then at least for one of $f_i (i = 0, 1)$, there are infinitely many x such that $K(x) \geq^+ f_i(x)$.
- Suppose $i = 1$, then f_1 is a Solovay function.
- For any sequence A , for all even number n , $K(A \upharpoonright n) \leq^+ 2n = f_1(n)$.



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Theorem

For any function f which goes to infinity, there exists a sequence A such that A is not infinitely often K-trivial but $\exists^\infty n \, K(A \upharpoonright n) \leq^+ K(n) + f(n)$.

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That's to say, among all right-c.e. functions which are upper bounds of K , if for any sequence A , A is infinitely often K-trivial iff $\exists^\infty n K(A \upharpoonright n) \leq^+ g(n)$, then g is a weak Solovay function.

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But whether all computable (right-c.e.) functions which make the equivalence true should be (weak) Solovay functions is still open.

Summary

Let g be any weak Solovay function, the following assertions are true.

- 1 $\sum_n 2^{-g(n)}$ is a Martin-Löf random real.
- 2 A sequence A is K-trivial iff $\forall n K(A \upharpoonright n) \leq^+ g(n)$.
- 3 A sequence A is Martin-Löf random iff $\forall n C(A \upharpoonright n) \geq^+ n - g(n)$.
- 4 A sequence A is weakly low for K, iff $\exists^\infty n K^A(n) \geq^+ g(n)$.
- 5 A sequence A is 2-random, iff $\exists^\infty n K(A \upharpoonright n) \geq^+ n + g(n)$.

What's more, among all right-c.e. functions the respective assertion is true exactly for the Solovay functions.

The no-gap phenomena

Theorem

Suppose f is a right-c.e. function, the following are equivalent:

- 1** $\forall x[K(x) \leq^+ f(x)];$
- 2** $\sum_n 2^{-f(n)} < \infty;$
- 3** *If A is K -trivial, then $\forall n K(A \upharpoonright n) \leq^+ f(n);$*
- 4** *If A is 1-random, then $\forall n C(A \upharpoonright n) \geq^+ n - f(n).$*
- 5** *If $\exists^\infty n K^A(n) \geq^+ f(n),$ then A is weakly low for $K;$*
- 6** *If $\exists^\infty n K(A \upharpoonright n) \geq^+ n + f(n),$ then A is 2-random;*

The no-gap phenomena

Theorem

Suppose f is a right-c.e. function, and f is an upper bound for K , the following are equivalent:

- 1 $\exists^\infty x [K(x) \geq^+ f(x)]$;
- 2 $\sum_n 2^{-f(n)}$ is 1-random;
- 3 If $\forall n K(A \upharpoonright n) \leq^+ f(n)$, then A is K -trivial;
- 4 If A is weakly low for K , then $\exists^\infty n K^A(n) \geq^+ f(n)$;
- 5 If A is 2-random, then $\exists^\infty n K(A \upharpoonright n) \geq^+ n + f(n)$;
- 6 If $\forall n C(A \upharpoonright n) \geq^+ n - f(n)$, then A is 1-random.

The no-gap phenomena

In the proof of our previous theorems, we proved the following so-called “no-gap” theorems:

No-gap

There is no function $h : \mathbb{N} \mapsto \mathbb{N}$ which tends to infinity and such that:

- 1 $C(A \upharpoonright n) \geq^+ n - K(n) - h(n) \implies A$ is Martin-Löf random;
- 2 $\exists^\infty n K(A \upharpoonright n) \leq^+ K(n) + h(n) \implies A$ is infinitely often K -trivial;

For K -triviality, George and Charlotte showed that there is no Δ_2^0 “gap”, but Martijn and George showed there does exist a Δ_3^0 “gap”.