# Monotonous betting strategies in warped casinos

by Nan Fang

Joint work with George Barmpalias and Andy Lewis-Pye

July 2018, Paris

A roulette system (strategy) has two components:

- the outcome where we bet
- the amount we bet

Many well-known strategies are single-sided:

- Martingale
- Grand Martingale
- D'Alembert
- Reverse D'Alembert
- Oscars Grind
- Laboucher or Cancellation System
- The Paroli

www.smartgaming.com/Articles/gambling_tips_top_10_roulette_systems.html

# Martingale strategy and D' Alembert system

| Coup Nr. | Pieces in use | Total loss in pieces |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 2 | 3 |
| 3 | 4 | 7 |
| 4 | 8 | 15 |
| 5 | 16 | 31 |
| 6 | 32 | 63 |
| 7 | 64 | 127 |
| 8 | 128 | 255 |
| 9 | 256 | 511 |
| 10 | 512 | 1023 |

| Coup | Use | Loss | Profit | Total |
|---|---|---|---|---|
| 1 | 1 | 1 | | -1 |
| 2 | 2 | 2 | | -3 |
| 3 | 3 | 3 | | -6 |
| 4 | 4 | 4 | | -10 |
| 5 | 5 | | 5 | -5 |
| 6 | 4 | | 4 | -1 |
| 7 | 3 | | 3 | +2 |
| 8 | 2 | | 2 | +4 |
| 9 | 1 | | 1 | +5 |

- provide certain desired profit x, with extremely high probability
- they work in (and in fact require) fair casinos.

# What's the catch?

Fairness means that on average, the expectation of profit is 0.

> The only way to get profit x with very high probability, is that with very small probability you lose a large amount, compared to x.

Other realistic considerations are:

- With high probability, you need to have access to considerable funds, for a considerable profit
- Upper limits set by the casino may block an aggressive strategy
- Slight advantages of the casino, such as the existence of 0 in the roulette

# How about biased casinos?

We ought to be able to do better in warped casinos.

Suppose that the frequency of 0s is $> 3/4$, infinitely often.

Then the following single-sided strategy guarantees you unbounded wealth:

- at each stage bet half of your capital on 0.

More generally:

If the relative frequency of 0s is $> q > 1/2$ infinitely often then the following single-sided strategy guarantees you unbounded wealth:

At each stage bet $(2q - 1)$ times your current capital, on 0.

# Modelling betting strategies

A stage in a game is determined by the series of previous outcomes.

A strategy is a function that maps any finite sequences of outcomes to the amount the player bets at that particular stage.

If $\mathrm{M}(\sigma)$ is the capital at $\sigma$ and we bet b on 0, then $\mathrm{b} \leq \mathrm{M}(\sigma)$ and

$$\mathrm{M}(\sigma * 0) = \mathrm{M}(\sigma) + \mathrm{b} \quad \text{and} \quad \mathrm{M}(\sigma * 1) = \mathrm{M}(\sigma) - \mathrm{b} \quad \text{so}$$

$$\mathrm{M}(\sigma) = \frac{\mathrm{M}(\sigma * 0) + \mathrm{M}(\sigma * 1)}{2}.$$

Conversely given M we can define the bet: $\mathrm{b}_{\mathrm{M}}(\sigma * \mathrm{i}) = \mathrm{M}(\sigma * \mathrm{i}) - \mathrm{M}(\sigma)$.

If I consider $\mathrm{M}_{\mathrm{s}}$, the capital at stage s, as a random variable in a fair game, then $(\mathrm{M}_{\mathrm{s}})$ is a martingale, relative to the outcome process $(\mathrm{I}_{\mathrm{s}})$:

$$\mathbb{E}(\mathrm{M}_{\mathrm{s}+1} \mid \mathrm{I}_{\mathrm{s}}) = \mathrm{M}_{\mathrm{s}}$$

Hence the probability that $\mathrm{M}_{\mathrm{s}} > 100 \cdot \mathrm{M}_0$ at some stage s is $< 1/100$.

# Success of strategies

The standard success measure is the condition:

$$\limsup_{n} M(X \restriction_n) = \infty$$

while a savings method shows that this is equivalent to

$$\lim_{n} M(X \restriction_n) = \infty$$

Exponential success means that:

$$\limsup_{n} \frac{M(X \restriction_n)}{\alpha^n} = \infty \qquad \text{for some } \alpha > 1.$$

Realistic considerations:

- infinitely divisible currency
- time to success
- computational restrictions
- inflation

## Biased Casinos

When can we say that a casino is biased?

In general, when there is a strategy that succeeds on its outcomes.

It is reasonable to assume that strategies are programmable.

In general, we restrict the strategies in a countable collection of functions.

Having a sequence $(\mathrm{N_i})$ of strategies with $\sum_i \mathrm{N_i}(\lambda) < \infty$, consider the:

Mixture strategy: $\sigma \mapsto \sum_i \mathrm{N_i}(\sigma)$ which is a

- composition of many strategies
- each using its own capital pool
- and betting independently of each other.

# Question

Can we always succeed in a biased casino with a single-sided strategy?

Can any successful strategy be replaced with a single-sided strategy?

For the programmable strategies the answer is yes.

For the mixtures of programmable strategies, the answer is no.

In fact, there exists a casino where:

- a programmable mixture strategy succeeds exponentially fast
- but no single-sided programmable mixture succeeds at all.

# Idea behind positive answer

A winning strategy must do well at least in one of the red or black bets.

> Every martingale is the product of a 0-sided martingale and a 1-sided martingale.

Also the component martingales are computable in the original martingale.

0-Strategy:

- bet on 0s as before but with modified amounts, and on 1s cancel the bets.
- On the next 0 bet as before, but proportionally relative to the ratio N/M on the previous 0-bet.
- So, scale according to the ratio of the N and M capitals at that point.

# Negative answer for mixtures

An elaborate construction of a casino which is:

- immune to (generalized) single-sided mixture strategies
- rigged by a mixture strategy which is successful exponentially fast.

Find more details of the argument in our paper.

# Single-sided martingales

A (super)martingale M is

- 0-sided if $M(\sigma * 0) \geq M(\sigma * 1)$ for all $\sigma$;
- 1-sided if $M(\sigma * 1) \geq M(\sigma * 0)$ for all $\sigma$.

We say that M is single-sided if it is either 0-sided or 1-sided.

Observations

- There exists an optimal 0-sided (1-sided) left-c.e. supermartingale.
- For any 0-sided martingale M there is a 0-sided martingale N such that for all sequences X $\limsup_n M(X \upharpoonright n) = \infty$ iff $\lim_n N(X \upharpoonright n) = \infty$.

# Single-sided martingales

**Lemma**

Every martingale $M$ is the product of a 0-sided martingale $N$ and a 1-sided martingale $T$. Moreover $N$, $T$ are computable from $M$.

**Corollary**

Given a computable martingale $M$, there exist a 0-sided martingale $N$ and a 1-sided martingale $T$ such that for each $X$ with $\limsup_n M(X \upharpoonright n) = \infty$ we have $\limsup_n N(X \upharpoonright n) = \infty$ or $\limsup_n T(X \upharpoonright n) = \infty$.

# Single-sided martingales

Theorem
There exists a real of effective Hausdorff dimension 1/2 such that no single-sided left-c.e. martingale succeeds on it.

Recall that the effective Hausdorff dimension of a real X is defined by

$$\mathrm{dim}(X) = \lim_{n} \frac{K(X \upharpoonright n)}{n}$$

and all Martin-Löf random real has Hausdorff dimension 1.

# proof idea

**Lemma (Hoeffding's inequality)**

Given a prediction function f, $1/2 < q < 1$ and n, the number of strings in $2^n$ with number of correct predictions more than qn is at most $r_q^{-n} \cdot 2^n$, where $r_q > 1$ is a constant depends only on q.

**Lemma**

For each $\epsilon \in (0, 1)$ there exists $m_0 > 0$ such that for all $m \geq m_0$ the number of strings in $2^m$ with the number of 0s (and hence, 1s) in $[(1 - \epsilon)m/2, (1 + \epsilon)m/2]$ is at least $(1 - \epsilon)2^m$.

## Kastergales

A kastergale is a pair $(\mathrm{f}, \mathrm{N_f})$ where $\mathrm{f}\colon 2^{<\omega} \mapsto \{0,1\}$ is a partial computable function and $\mathrm{N_f}$ is a left-c.e. supermartingale with monotone approximation $\mathrm{N_f}[\mathrm{s}]$ and the property that:

- $\mathrm{f}(\sigma)[\mathrm{s}] \uparrow \Rightarrow \mathrm{N}(\sigma * 0)[\mathrm{s}] = \mathrm{N}(\sigma * 1)[\mathrm{s}]$;
- $\mathrm{f}(\sigma)[\mathrm{s}] \downarrow \Rightarrow \mathrm{N}(\sigma * \mathrm{f}(\sigma))[\mathrm{s}] = \mathrm{N}(\sigma * (1 - \mathrm{f}(\sigma)))[\mathrm{s}]$.

Theorem

There exists a real of effective Hausdorff dimension 1/2 such that no kastergale succeeds on it.

# Related literature


- Ron Peretz
  Effective martingales with restricted wagers
  Information and Computation 245 (2015) 152–164

- Gilad Bavly and Ron Peretz
  How to gamble against all odds
  Games and Economic Behavior 94 (2015) 157–168

- Adam Chalcraft, Randall Dougherty, Chris Freiling, Jason Teutsch
  How to build a probability-free casino
  Information and Computation 211 (2012) 160–164

- Jason Teutsch
  A savings paradox for integer-valued gambling strategies
  Int J Game Theory (2014) 43:145–151