Granularity of wagers in games and the (im)possibility of savings

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betting strategies

- You are in a casino, betting your money on binary outcomes.
- Wager is the amount you bet for the chosen outcome each round.
- If you win, you double your wager.
- If you lose, the casino takes your wager.
- You may occasionally save some money by withdrawing it forever from the casino.

As a good player, along an infinite outcome sequence, you may

- reach unbounded profit (success strategy);
- or save unbounded profit (savings strategy).

As realistic restrictions, the casino may require each wager to be

- a multiple of some betting unit;
- linearly bounded by the betting unit.

martingales as betting strategies

A betting strategy is presented by a (super)martingale which records the capital after each stage, and which is a function $2^{<\omega} \mapsto \mathbb{R}^{0+}$ such that

$$M(\sigma) \geq \frac{M(\sigma^\frown 0) + M(\sigma^\frown 1)}{2}.$$

- wager on outcome i at stage σ:

$$w_M(\sigma^\frown i) = \frac{M(\sigma^\frown i) - M(\sigma^\frown(1-i))}{2}.$$

- marginal savings at stage σ:

$$M^*(\sigma) = M(\sigma) - \frac{M(\sigma^\frown 0) + M(\sigma^\frown 1)}{2}.$$

- cover of M:

$$\widehat{M}(\sigma) = M(\lambda) + \sum_{\tau \leq \sigma} w_M(\tau).$$

- (accumulated) savings of M:

$$S_M(\sigma) = \sum_{\tau < \sigma} M^*(\tau).$$

$$M(\sigma^\frown i) = M(\sigma) - M^*(\sigma) + w_M(\sigma^\frown i),$$

$$S_M(\sigma) = \widehat{M}(\sigma) - M(\sigma).$$

success of betting strategies

For a (super)martingale M and a sequence X, we say M

- succeeds on X, if $\limsup_{n\to\infty} M(X \upharpoonright n) = \infty$;
- strongly succeeds on X, if $\lim_{n\to\infty} M(X \upharpoonright n) = \infty$;
- successfully saves on X, if $\lim_{n\to\infty} S_M(X \upharpoonright n) = \infty$.

$$\mathrm{Succ}(M) := \{X \colon M \text{ succeeds on } X\}$$

$$\mathrm{SSucc}(M) := \{X \colon M \text{ strongly succeeds on } X\}$$

$$\mathrm{Save}(M) := \{X \colon M \text{ successfully saves on } X\}$$

For a class $\mathcal{C}$ of (super)martingales,

$$\mathrm{Succ}[\mathcal{C}] := \bigcup_{M \in \mathcal{C}} \mathrm{Succ}(M).$$

$\mathrm{SSucc}[\mathcal{C}]$ and $\mathrm{Save}[\mathcal{C}]$ are defined similarly.

savings trick vs. savings paradox

savings trick (folklore)

From any supermartingale M one can construct a supermartingale N, such that $\mathrm{Succ}(\mathrm{M}) \subseteq \mathrm{Save}(\mathrm{N})$.

savings paradox (Jason Teutsch)

If only integer-valued wager is allowed, there exists a supermartingale M such that for any countable class $\mathcal{C}$ of supermartingales $\mathrm{Succ}(\mathrm{M}) \not\subseteq \mathrm{Save}[\mathcal{C}]$.

granular betting strategies

Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a nondecreasing function.

- A function $f\colon 2^{<\omega} \mapsto \mathbb{Q}$ is g-granular if $f(\sigma)$ is an integer multiple of $2^{-g(|\sigma|)}$ for any $\sigma \in 2^{<\omega}$;
- A (super)martingale M is g-granular if its wager w_M is a g-granular function.

We distinguish between

- fine granularity (where $\sum_n 2^{-g(n)} < \infty$) and
- coarse granularity (where $\sum_n 2^{-g(n)} = \infty$).

savings trick for fine granularity

savings trick for fine granularity

Let g be of fine granularity. Given any supermartingale M, there exists a g-granular supermartingale N computable from M, g such that $\mathrm{Succ}(M) \subseteq \mathrm{Save}(N)$.

- Assume the capital of M at each stage is larger than 1.
- N starts with sufficient large initial capital.
- At each stage the wager of N is chosen proportionally to the wager of M, while rounded to the closest granular value.
- Whenever the capital of M doubles, N saves 1.

□ Our construction ensures that at any stage $N/M > 1$, i.e. $N \geq M$. N is a well-defined supermartingale.

□ If M achieves infinite capital along some sequence, its capital doubles infinitely many times, then N saves infinite capital.

(weak) savings paradox for coarse granularity

weak savings paradox for coarse granularity

Let g be of coarse granularity. There exists a g-granular martingale M computable from g such that for any g-granular supermartingale T it holds that $\mathrm{Succ}(M) \not\subseteq \mathrm{Save}(T)$.

- The martingale M starts with capital 1 and always bets one granule on outcome '1' if possible.
- Given T, we define a sequence X such that M succeeds and T does not successfully save.
- For a stage n where $X \upharpoonright n$ is already defined, to choose the next bit of X we monitor the Euclidean division:

$$\begin{aligned} & T(X \upharpoonright n) = q \cdot M(X \upharpoonright n) + r \\ \to \quad & T(X \upharpoonright n^\frown i) = q' \cdot M(X \upharpoonright n^\frown i) + r'. \end{aligned}$$

- We choose $X(n) = i$ such that either $q' < q$ or $q' = q$ and $r' \leq r$.

(weak) savings paradox for coarse granularity

- We choose $X(n) = i$ such that either $q' < q$ or $q' = q$ and $r' \leq r$.

Failed saving of T along X:

- q as an integer will reach a final value at some stage s.
- After stage s, the remainder r will be non-increasing.
- After stage s, savings of T are taken from r, so savings of T is bounded.

Success of M along X:

- Then after stage s, the total loss of M with outcome '0' is bounded by the total loss of r, which is finite.
- On the other hand, since g is coarse granularity, the total increase of M with outcome '1' is unbounded.

timid betting strategies

Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a nondecreasing function.

- A function $f\colon 2^{<\omega} \mapsto \mathbb{Q}$ is g-timid if it is g-granular and there is a constant c such that $f(\sigma) \leq c \cdot 2^{-g(|\sigma|)}$ for all $\sigma \in 2^{<\omega}$.
- A (super)martingale M is g-timid if its wager w_M is a g-timid function.

Note that in the previous proof the martingale we constructed is actually g-timid.

savings paradox for coarse timid strategies

savings paradox for coarse timid strategies

Let g be coarse granularity. There exists a g-timid martingale M computable from g such that for any countable class $\mathcal{C}$ of g-timid supermartingales $\mathrm{Succ}(M) \nsubseteq \mathrm{Save}[\mathcal{C}]$.

- The martingale M is defined as before.
- Given g-timid supermartingales $\{T_i\}_{i\in\omega}$, we define a sequence X such that M succeeds and none of $\{T_i\}_{i\in\omega}$ successfully saves.
- For a stage n where $X \upharpoonright n$ is already defined, to choose the next bit of X we monitor the tower of Euclidean division:

$$\begin{aligned} & T_i(X \upharpoonright n) = q_i \cdot M(X \upharpoonright n) + r_i \\ \rightarrow \quad & T_i(X \upharpoonright n^\frown i) = q_i' \cdot M(X \upharpoonright n^\frown i) + r_i'. \end{aligned}$$

- We choose $X(n) = i$ such that either $\langle q_0, q_1, \cdots, q_\ell\rangle < \langle q_0', q_1', \cdots, q_\ell'\rangle$ or $q_i' = q_i$ and $r_i' \leq r_i$ for all $0 \leq i \leq \ell$, where ℓ is the largest index of the supermartingale considered at the stage.

Let g be any non-decreasing function which goes to infinity. For any g-timid supermartingale M, there is a countable class $\mathcal{C}$ of g-granular supermartingales such that $\mathrm{Succ}(\mathrm{M}) \subseteq \mathrm{Save}[\mathcal{C}]$. Moreover, all the martingales in $\mathcal{C}$ are computable from M, g.

Given a g-timid supermartingale M, we construct a main supermartingale T and a series $\{\mathrm{N}_\rho\}_{\rho \in \mathrm{I}}$ of backup supermartingales.

- Every sequence is divided into neutral intervals and cycle-intervals.
- At neutral intervals both T and N_ρ follows M identically.
- During cycles, T doubles the wager of M on the same outcome, while N_ρ multiplies the wager of M by a large negative coefficient.
- At the end of a cycle at stage σ,
  - if T gains extra 1, we let T save 1, close N_ρ and initiate a new backup supermartingale N_σ;
  - if N_ρ gains extra 1, we let N_ρ save 1, and let it continue serve as the active backup supermartingale.

weakness of timid martingales

- By the g-timidness of M, we make sure there is always space for starting a new cycle.
- We also make sure that if $\limsup_n M(X \upharpoonright n) = \infty$ then one of the following will happen.
  - infinitely many backup supermartingales initiated and closed along X: in this case T successfully saves;
  - there is a last supermartingale N_ρ initiated along X, which is never closed and generates infinitely many cycles along X, each of them ending in failure for T: in this case N_ρ successfully saves.

a further question for granular martingales

Open question:

Let g be coarse granularity. Given any g-granular supermartingale M, is there a countable class $\mathcal{C}$ of g-granular supermartingales such that $\mathrm{Succ}(\mathrm{M}) \subseteq \mathrm{Save}[\mathcal{C}]$?

We want to know whether the randomnesses induced by these two different notions of success for the class of coarse granular computable supermartingales coincide.

References – Literature

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  A savings paradox for integer-valued gambling strategies
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- Ron Peretz
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Thanks for your attention!