Betting with preference on outcomes

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effective (super)martingales

A betting strategy is presented by a (super)martingale which records the capital after each stage, and which is a function $2^{<\omega} \mapsto \mathbb{R}^{0+}$ such that

$$\mathrm{M}(\sigma) \geq \frac{\mathrm{M}(\sigma^\frown 0) + \mathrm{M}(\sigma^\frown 1)}{2}.$$

We let $\mathrm{Succ}(\mathrm{M}) := \{\mathrm{X} \colon \limsup_{\mathrm{n}\to\infty} \mathrm{M}(\mathrm{X} \upharpoonright \mathrm{n}) = \infty\}$.

A (super)martingale is

- computable/left-c.e. if it is a computable/left-c.e. function.
- strongly left-c.e. if it is the sum of a family of uniformly computable (super)martingale.

Every left-c.e. martingale is strongly left-c.e.

monotonous (super)martingales

A (super)martingale M is

- 0-sided if $\mathrm{w_M}(\sigma^\frown 0) \geq 0$ for all σ;
- 1-sided if $\mathrm{w_M}(\sigma^\frown 1) \geq 0$ for all σ;
- single-sided if it is either 0-sided or 1-sided;
- separable if it is the sum of a 0-sided and a 1-sided martingale.

Many well-known strategies for roulette are single-sided, for example, the infamous "martingale".

decomposition of a supermartingale

Every martingale M is the product of a 0-sided martingale N and a 1-sided martingale T. Moreover N, T are computable from M.

- Let $c_{\mathrm{N}}(\lambda) = c_{\mathrm{T}}(\lambda) = \sqrt{\mathrm{M}(\lambda)}$.
- For all $\sigma \in 2^{<\omega}$ and $\mathrm{i} \in \{0,1\}$, let $c_{\mathrm{M}}(\sigma^\frown \mathrm{i}) = \begin{cases} \frac{\mathrm{M}(\sigma^\frown \mathrm{i})}{\mathrm{M}(\sigma)} & \text{if } \mathrm{M}(\sigma) > 0, \\ 1 & \text{otherwise}, \end{cases}$

$$\begin{array}{llll} c_{\mathrm{N}}(\sigma^\frown \mathrm{i}) = c_{\mathrm{M}}(\sigma^\frown \mathrm{i}) & \text{and} & c_{\mathrm{T}}(\sigma^\frown \mathrm{i}) = 1 & \text{if } c_{\mathrm{M}}(\sigma^\frown 0) > 1, \\ c_{\mathrm{N}}(\sigma^\frown \mathrm{i}) = 1 & \text{and} & c_{\mathrm{T}}(\sigma^\frown \mathrm{i}) = c_{\mathrm{M}}(\sigma^\frown \mathrm{i}) & \text{if } c_{\mathrm{M}}(\sigma^\frown 0) \leq 1. \end{array}$$

- Let $\mathrm{N}(\sigma) = \prod_{\tau \preceq \sigma} c_{\mathrm{N}}(\tau)$ and $\mathrm{T}(\sigma) = \prod_{\tau \preceq \sigma} c_{\mathrm{T}}(\tau)$.
- Then $\mathrm{M}(\sigma) = \mathrm{N}(\sigma) \cdot \mathrm{T}(\sigma)$ for all $\sigma \in 2^{\omega}$.

computable separable supermartingales

Given a computable martingale M, there exist a computable 0-sided martingale N and a computable 1-sided martingale T such that $\mathrm{Succ}(M) \subseteq \mathrm{Succ}(N) \cup \mathrm{Succ}(T)$. In particular, for the computable separable martingale $S = N + T$, $\mathrm{Succ}(M) \subseteq \mathrm{Succ}(S)$.

- We say a sequence is computably single-sided random if there is no computable single-sided martingale succeed on it.
- A sequence is computably random if there is no computable martingale succeed on it.

The above result shows that these two randomness notion coincide.

strongly left-c.e. separable supermartingales

A single-sided/separable (super)martingale is

- strongly left-c.e. if it is the sum of a family of uniformly computable single-sided (super)martingale.

There is an optimal strongly left-c.e. 0-sided/1-sided/separable supermartingale.

success on sequences with dim < 1/2

There exists a strongly left-c.e. separable martingale M which succeeds on all sequences with dimension less than $1/2$.

One characterizations of effective Hausdorff dimension is that

$$\dim(X) = \sup\{s \mid X \text{ is weakly } s\text{-random}\} = \liminf_n \frac{K(X \upharpoonright n)}{n}.$$

- Given $s \in (0,1)$, X is said to be weakly s-random if for all s-tests $\{V_i\}_{i\in\omega}$, X has a prefix only in finitely many V_i.
- An s-test is a family $\{V_i\}_{i\in\omega}$ of uniformly c.e. sets of strings such that $\sum_{\sigma\in V_i} 2^{-s|\sigma|} < 2^{-i}$ for each i.

The proof idea is to construct the strongly left-c.e. separable martingale from a universal $1/2$-test.

failure on a sequence with dim 1/2

For any strongly left-c.e. separable supermartingale M, there exists a sequence X with dimension 1/2 on which M does not succeed.

Proof idea: Given a strongly left-c.e. separable supermartingale M, we construct a series of strings $\{\sigma_n\}_{n\in\omega}$ such that

- $\sigma_0 = \lambda$, $\sigma_n \prec \sigma_{n+1}$;
- $\forall n\ \hat{M}(\sigma_n) \le 1 - 2^{-n-1}$, where $\hat{M}(\rho) = \max\{M(\tau) : \tau \preceq \rho\}$;
- whenever σ_n is newly (re)defined, enumerating a code of length $(1/2 + 2^{-n-1}) \cdot |\sigma_n|$ into a prefix-free machine V.

Thus, if we let $X = \lim_n \sigma_n$, it holds that

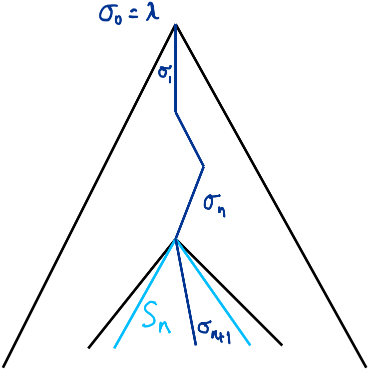
- ☐ $\limsup_n M(X \upharpoonright n) \le 1$
- ☐ $\liminf_n \frac{K(X\upharpoonright n)}{n} \le \frac{K(\sigma_n)}{|\sigma_n|} \le 1/2$

construction of σ_{n+1}

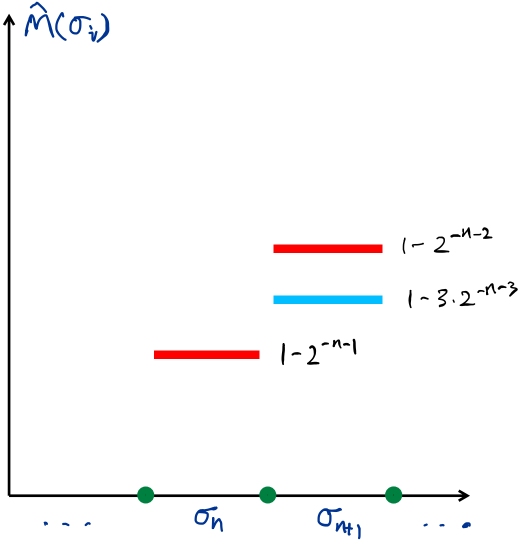
Supposing inductively that σ_n has been determined, then $\hat{M}_s(\sigma_n) \leq 1 - 2^{-n-1}$. Now define σ_{n+1} such that $\hat{M}(\sigma_{n+1}) \leq 1 - 2^{-n-2}$.

- First, by Hoeffding's inequality find a set S_n of strings extending σ_n of extra length ℓ_n, such that all strings in S_n have almost equal 0s and 1s among the bits extending σ_n and $|S_n|$ is close to 2^{ℓ_n}.
- Find a candidate τ for σ_{n+1} in S_n such that $\hat{M}_s(\tau) \leq 1 - 3 \cdot 2^{-n-3}$. By Kolmogorov's inequality, such candidate can always be found.
- If $\hat{M}_t(\tau) > 1 - 2^{-n-2}$ later, we say τ is bad and find another candidate. As long as σ_n is still good, the candidate can always be found.

- ☐ If τ turns out to be bad, then $\hat{M}_t(\tau) - \hat{M}_s(\tau) \geq 2^{-n-3}$.
- ☐ By the strongly left-c.e. monotonousness of M, it follows that $\hat{M}_t(\sigma_n) - \hat{M}_s(\sigma_n) \geq 2^{-n-3-\ell_n/2}$.
- ☐ Then the approximation to σ_{n+1} may change at most $2^{n+3+\ell_n/2}$ many times for fixed σ_n. In this way, the measure of codes enumerated into V for σ_{n+1} can be limited.



$\sigma_0 = \lambda$
σ_1
σ_n
S_n
σ_{n+1}



$\hat{M}(\sigma_i)$
$1-2^{-n-2}$
$1-3\cdot 2^{-n-3}$
$1-2^{-n-1}$
...
σ_n
σ_{n+1}
...

generalization to decidably-sided supermartingales

- A prediction function f is a function from 2^ω to $\{0,1\}$.
- Given a prediction function f, a (super)martingale M is f-sided if for all σ, $M(\sigma^\frown i) > M(\sigma^\frown(1-i))$ only if $f(\sigma)\downarrow = i$.
- A (super)martingale M is decidably-sided if it is f-sided for a computable prediction function f.

A modification of the proof for separable supermartingales gives us the following generalized result.

> There exists a sequence with dimension 1/2 such that no strongly left-c.e. decidably-sided supermartingale succeeds on it.

Limitations of our methods

There are two limitations for our methods.

1. It applies only to strongly left-c.e. supermartingales, not left-c.e. supermartingales.
2. It requires the candidates sets to be computable.

LS	class of all left-c.e. supermartingales
dLS	class of all left-c.e. decidably-sided supermartingales
dSLS	class of all strongly left-c.e. decidably-sided supermartingales

We have already shown that $\mathrm{Succ}[\mathsf{dSLS}] \neq \mathrm{Succ}[\mathsf{LS}]$ and obviously,

$$\mathrm{Succ}[\mathsf{dSLS}] \subseteq \mathrm{Succ}[\mathsf{dLS}] \subseteq \mathrm{Succ}[\mathsf{LS}].$$

Because of the first limitation, our method cannot tell whether which of the inclusion is strict.

Kastermans problems

- A (super)martingale M is partially decidably-sided if it is f-sided for a partial computable prediction function f.
- A a Kastergale is a left-c.e. partially decidably-sided supermartingale.
- A a Hitchgale is a strongly left-c.e. partially decidably-sided supermartingale.

Open Question (Kastermans)

If a left-c.e. supermartingale succeeds on X, does there exist a Kastergale which succeeds on X as well?

Open Question (Hitchcock)

If a left-c.e. supermartingale succeeds on X, does there exist a Hitchgale which succeeds on X as well?

Our work relates to these two open questions, while because of the second limitation, our method cannot solve them either.

Thank you for your attention!