Granularity and Timidness of Betting Strategies

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betting strategies

- You are in a casino, betting your money on binary outcomes.
- Wager is the amount you bet for the chosen outcome each round.
- If you win, you double your wager.
- If you lose, the casino takes your wager.
- You may occasionally save some money by withdrawing it forever from the casino.

As a good player, along an infinite outcome sequence, you may

- reach unbounded profit (success strategy);
- or save unbounded profit (savings strategy).

As realistic restrictions, the casino may require each wager to be

- a multiple of some betting unit;
- linearly bounded by the betting unit.

savings trick vs. savings paradox

savings trick (folklore)

From any strategy M one can construct a strategy N, such that if M reaches unbounded profit along X then N saves unbounded profit along X.

savings paradox (Teutsch)

If only integer-valued wager is allowed, there exists a strategy M such that for any countable collection of strategies $\{N_i\}_{i \in \omega}$, there is a sequence X such that M reaches unbounded profit along X while none of $\{N_i\}_{i \in \omega}$ can save unbounded profit along X.

martingales as betting strategies

A betting strategy is presented by a (super)martingale which records the capital after each stage, and which is a function $2^{<\omega} \mapsto \mathbb{R}^{0+}$ such that

$$M(\sigma) \geq \frac{M(\sigma^\frown 0) + M(\sigma^\frown 1)}{2}.$$

- wager on outcome i at stage σ:

$$w_M(\sigma^\frown i) = \frac{M(\sigma^\frown i) - M(\sigma^\frown (1-i))}{2}.$$

- marginal saving at stage σ:

$$M^*(\sigma) = M(\sigma) - \frac{M(\sigma^\frown 0) + M(\sigma^\frown 1)}{2}.$$

- cover of M:

$$\widehat{M}(\sigma) = M(\lambda) + \sum_{\tau \leq \sigma} w_M(\tau).$$

- (accumulated) saving of M:

$$S_M(\sigma) = \sum_{\tau < \sigma} M^*(\tau).$$

$$M(\sigma^\frown i) = M(\sigma) - M^*(\sigma) + w_M(\sigma^\frown i),$$

$$S_M(\sigma) = \widehat{M}(\sigma) - M(\sigma).$$

success of betting strategies

For a (super)martingale M and a sequence X, we say M

- succeeds on X, if $\limsup_{n\to\infty} M(X \upharpoonright n) = \infty$;
- strongly succeeds on X, if $\lim_{n\to\infty} M(X \upharpoonright n) = \infty$;
- successfully saves on X, if $\lim_{n\to\infty} S_M(X \upharpoonright n) = \infty$.

$$\mathrm{Succ}(M) := \{X \colon M \text{ succeeds on } X\}$$

$$\mathrm{SSucc}(M) := \{X \colon M \text{ strongly succeeds on } X\}$$

$$\mathrm{Save}(M) := \{X \colon M \text{ successfully saves on } X\}$$

For a class $\mathcal{C}$ of (super)martingales,

$$\mathrm{Succ}[\mathcal{C}] := \bigcup_{M \in \mathcal{C}} \mathrm{Succ}(M).$$

$\mathrm{SSucc}[\mathcal{C}]$ and $\mathrm{Save}[\mathcal{C}]$ are defined similarly.

granular strategies and timid strategies

Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a nondecreasing function. A function $f\colon 2^{<\omega} \mapsto \mathbb{Q}$ is

- g-granular if $f(\sigma) \cdot 2^{g(|\sigma|)} \in \mathbb{N}$ for any $\sigma \in 2^{<\omega}$;
- g-timid if it is g-granular and there is a constant c such that $f(\sigma) \leq c \cdot 2^{-g(|\sigma|)}$ for all $\sigma \in 2^{<\omega}$.

A (super)martingale M is g-granular or g-timid if its wager w_M is a g-granular or g-timid function, respectively.

We distinguish between

- fine granularity (where $\sum_n 2^{-g(n)} < \infty$) and
- coarse granularity (where $\sum_n 2^{-g(n)} = \infty$).

savings trick for fine granularity

savings trick for fine granularity

Let g be fine granularity. Given any supermartingale M, there exists a g-granular supermartingale N computable from M, g such that $\mathrm{Succ}(M) \subseteq \mathrm{Save}(N)$.

- Assume the capital of M at each stage is larger than 1.
- N starts with initial capital $N(\lambda) = (\sum_n 2^{-g(n)} + 2) \cdot M(\lambda)$.
- At each stage the wager of N is chosen proportionally to the wager of M, while rounded to the closest granular value.
- Whenever the capital of M doubles, N saves 1.

- □ At each stage the loss of the ratio N/M is less than one granule.
- □ At the i-th saving step, the loss of the ratio N/M is less than 2^{-i}.
- □ Thus, at any stage the total loss of the ratio is less than $\sum_n 2^{-g(n)} + 1$. Then $N/M > 1$, i.e. $N \geq M$. N is a well-defined supermartingale.
- □ If M achieves infinite capital along some sequence, its capital doubles infinitely many times, then N saves infinite capital.

(weak) savings paradox for coarse granularity

weak savings paradox for coarse granularity

Let g be coarse granularity. There exists a g-granular martingale M computable from g such that for any g-granular supermartingale T it holds that $\mathrm{Succ}(M) \setminus \mathrm{Save}(T) \neq \emptyset$.

The martingale M starts with capital 1 and always bets one granule on outcome '1' if possible.

Task: Given T, define a sequence X such that M succeeds and T does not successfully save.

For a stage n where $X \upharpoonright n$ is already defined and we want to choose the next of X, consider the following random variables along X:

- m, t are the capitals of M, T before the current round.
- p, w are the wager of M, T on outcome '1'.
- m', t' are the capitals of M, T before the next round.

(weak) savings paradox for coarse granularity

Monitor the division: $t = q \cdot m + r \quad \to \quad t' = q' \cdot m' + r'$

Choose the outcome so that either $q' < q$ or $q' = q$ and $r' \leq r$.

$$X(n) = \begin{cases} 1 & \text{if } w \leq q \cdot p, \\ 0 & \text{if } w > q \cdot p. \end{cases}$$

Failed saving of T along X:

- q as an integer will reach a final value at some stage s.
- After stage s, the remainder r will be non-increasing.
- After stage s, savings of T are taken from r, so savings of T is bounded.

Success of M along X:

- After stage s, if ‘0’ is chosen then $r - r' \geq p = m - m'$.
- Then after stage s, the total loss of M with outcome ‘0’ is finite.
- Since $\sum_{n>s} p_n = \infty$, the total increase of M with outcome ‘1’ is unbounded.

savings paradox for coarse timid strategies

We notice that, in the previous proof the martingale we constructed is actually g-timid.

savings paradox for coarse timid strategies

Let g be coarse granularity. There exists a g-timid martingale M computable from g such that for any countable class $\mathcal{C}$ of g-timid supermartingales $\mathrm{Succ}(M) \setminus \mathrm{Save}[\mathcal{C}] \neq \emptyset$.

The martingale M is defined as before.

Task: Given g-timid supermartingales $\{T_i\}_{i\in\omega}$, define a sequence X such that M succeeds and none of $\{T_i\}_{i\in\omega}$ successfully saves.

For a stage n where $X \upharpoonright n$ is already defined and we want to choose the next of X, consider the following random variables along X:

- m, t_i are the capitals of M, T_i before the current round.
- p, w_i are the wager of M, T_i on outcome '1'.
- m', t_i' are the capitals of M, T_i before the next round.

savings paradox for coarse timid strategies

Monitor the tower of divisions:

$$\begin{cases} m_i = m_{i-1} - r_{i-1} \\ t_i + c_i \cdot r_{i-1} = q_i \cdot m_i + r_i \end{cases} \rightarrow \begin{cases} m'_i = m'_{i-1} - r'_{i-1} \\ t'_i + c_i \cdot r_{i-1} = q'_i \cdot m'_i + r'_i \end{cases}$$

Where $m_{-1} = m$, $r_{-1} = 0$ and c_i is the constant such that $w_i \leq (c_i + 1) \cdot p_i$.

- T_i is taken into consideration only if M has ever been larger than $i + 1$.
- Suppose k is the largest index i such that T_i is taken into consideration at the current stage.
- Guaranteed by the timidness, we can choose the outcome so that for all $\ell \leq k$ either $\langle q_0, q_1, \cdots, q_\ell \rangle < \langle q'_0, q'_1, \cdots, q'_\ell \rangle$ or $q'_i = q_i$ and $r'_i \leq r_i$ for all $0 \leq i \leq \ell$.

$$X(n) = \begin{cases} 0 & \text{if } w_i > q_i \cdot p \text{, where } i \in [0, k] \\ & \text{ is the least such that } w_i \neq q_i \cdot p, \\ 1 & \text{otherwise.} \end{cases}$$

savings paradox for coarse timid strategies

Failed saving of each T_i along X:

- After T_i has been taken into consideration, $\langle q_0, q_1, \cdots, q_i \rangle$ is an $i+1$-tuple of integer. So it will reach a final value at some stage s.
- After stage s, the remainder r_i will be non-increasing.
- After stage s, savings of T_i are taken from r_i, so savings of T_i is bounded.

Success of M along X:

- After stage s where $\langle q_0, q_1, \cdots, q_k \rangle$ reaches a final value, if outcome '0' is chosen then $r_i - r_i' \geq p = m - m'$ for some $i \leq k$.
- Then after stage s, the total loss of M with outcome '0' is finite.
- Since $\sum_{n>s} p_n = \infty$, the total increase of M with outcome '1' is unbounded.
- Thus, for every i, the value of M reaches $i+1$ at some stage, and T_i will be taken into consideration from then on.

weakness of timid martingales

Let g be any non-decreasing function which goes to infinity. For any g-timid supermartingale M, there is a countable class $\mathcal{C}$ of g-granular supermartingales such that $\mathrm{Succ}(\mathrm{M}) \subseteq \mathrm{Save}[\mathcal{C}]$. Moreover, all the martingales in $\mathcal{C}$ are computable from M, g.

Given a g-timid supermartingale M, we construct a main supermartingale T and a series $\{\mathrm{N}_\rho\}_{\rho \in \mathrm{I}}$ of backup supermartingales.

- At every stage there is exactly one active backup supermartingale N_ρ.
- Every sequence is divided into neutral intervals and cycle-intervals.
- At neutral intervals both T and N_ρ follows M identically.
- During cycles, T doubles the wager of M on the same outcome, while N_ρ multiplies the wager of M by a large negative coefficient.
- At the end of a cycle at stage σ, if T gains 1 extra dollar, we let T save 1, close N_ρ and initiate a new backup supermartingale N_σ.
- At the end of a cycle at stage σ, if N_ρ gains 1 extra dollar, we let N_ρ save 1, and let it continue serve as the active backup supermartingale.

weakness of timid martingales

- By the g-timidness of M, we make sure there is always space for starting a new cycle.
- We also make sure that if $\limsup_n M(X \upharpoonright n) = \infty$ then one of the following will happen.

  - infinitely many backup supermartingales initiated and closed along X: in this case T successfully saves;
  - there is a last supermartingale N_ρ initiated along X, which is never closed and generates infinitely many cycles along X, each of them ending in failure for T: in this case N_ρ successfully saves.

a further question for granular martingales

Open question:

Let g be coarse granularity. Given any g-granular supermartingale M, is there a countable class $\mathcal{C}$ of g-granular supermartingales such that $\mathrm{Succ}(\mathrm{M}) \subseteq \mathrm{Save}[\mathcal{C}]$?

We want to know whether the randomnesses induced by these two different notions of success for the class of coarse granular computable supermartingales coincide.