A study on granular martingales

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Betting strategies

- You are in a casino, betting your money on binary outcomes.
- Each round, you bet some wager on your guessed outcome.
- If you win, you double your wager.
- If you lose, the casino takes your wager.
- You may occasionally save/consume some money by withdrawing it forever from the casino.

As a *good* player, along with an infinite outcome sequence, you may

- reach unbounded profit (winning strategy)
- or save unbounded profit (saving strategy).

Formulation of betting strategies

A betting strategy is presented by a (super)martingale which records the capital after each stage, and which is a function $M\colon 2^{<\omega} \mapsto \mathbb{R}^{0+}$ such that

$$M(\sigma) \geq \frac{M(\sigma^\frown 0) + M(\sigma^\frown 1)}{2}.$$

- wager on i at stage σ:

$$w_M(\sigma^\frown i) = \frac{M(\sigma^\frown i) - M(\sigma^\frown(1-i))}{2}.$$

- guessed outcome is i if:

$$w_M(\sigma^\frown i) > 0.$$

- marginal savings at stage σ:

$$M^*(\sigma) = M(\sigma) - \frac{M(\sigma^\frown 0) + M(\sigma^\frown 1)}{2}.$$

- (accumulated) savings of M:

$$S_M(\sigma) = \sum_{\tau < \sigma} M^*(\tau).$$

- M succeeds on X, if $\limsup_{n\to\infty} M(X \upharpoonright n) = \infty$;
- M successfully saves on X, if $\lim_{n\to\infty} S_M(X \upharpoonright n) = \infty$.

Properties of betting strategies

If there is no restriction on the wager, a winning strategy always has the following two properties.

1. "win with small initial capital": Given a martingale A one can construct a martingale B with arbitrarily small initial capital, such that B succeeds on any sequence which A succeeds on.

2. "win by saving": Given a martingale A one can construct a martingale B, such that B successfully saves on any sequence which A succeeds on.
   (known as "savings trick")

Both properties can be easily proved with proportional betting strategy.

Restricting the wagers

As realistic restrictions, the casino may require each wager to be

- a multiple of some betting unit;
- linearly bounded by the betting unit.

If such restrictions are imposed to the wagers, the proportional betting strategy may be hindered.

Theorem ("savings paradox", [Teutsch])

There exists an integer-valued supermartingale M such that for any countable class C of integer-valued supermartingales one can always find a sequence on which M succeeds while none of C could successfully save.

Granular betting strategies

We consider a more general case where the betting unit shrinks over stages.

Definition (granular martingale)

Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a nondecreasing function.

- A function $f\colon 2^{<\omega} \mapsto \mathbb{Q}$ is g-granular if $f(\sigma)$ is an integer multiple of $2^{-g(|\sigma|)}$ for any $\sigma \in 2^{<\omega}$.
- A (super)martingale M is g-granular if its wager w_M is a g-granular function.

We study the two before mentioned properties and distinguish between

- fine granularity: g is a fast order.
- coarse granularity: g is constant or a slow order.

Fine granularity and initial capital

Theorem ([Barmpalias, Fang])

Let g be of fine granularity. Given any $\epsilon > 0$, from any g-granular supermartingale M one can construct a g-granular martingale N with initial capital ϵ such that N succeeds on any sequence M succeeds on.

In this case, it is still possible to bet proportionally with careful attention.

Coarse granularity and initial capital

Theorem ([Barmpalias, Fang])

Let g be of coarse granularity. For any $l \in \mathbb{N}^+$ there exists a g-granular martingale M with initial capital l such that for any g-granular supermartingale N with initial capital less than l, one can find a sequence on which M succeeds while N fails.

- Let M be the martingale that always bet one granule on side '1' if possible.
- For any martingale N, let X be the sequence such that every bit of it is opposite to the guessed outcomes according to N.
- Then N will fail on X, while M will succeed on it.

Granularity and initial capital

Theorem ([Barmpalias, Fang])

Let g be of coarse granularity. Given any $\epsilon \geq 2^{-g(n)}$ for some $n \in \mathbb{N}$, from any g-granular supermartingale M one can construct a class C of countable g-granular martingales with initial capital ϵ such that C succeeds on any sequence M succeeds on.

	fine granularity	coarse granularity
"win with small initial capital" for a single strategy	$\checkmark$	$\times$
"win with small initial capital" for a class of countable strategies	$\checkmark$	$\checkmark$

Fine granularity and saving strategies

Theorem ([Barmpalias, Fang])

Let g be of fine granularity. From any supermartingale M one can construct a g-granular supermartingale N such that N successfully saves on every sequence on which M succeeds.

In this case, it is still possible to bet proportionally with careful attention.

Coarse granularity and saving strategies

Theorem ([Barmpalias, Fang])

Let g be of coarse granularity. There exists a g-granular martingale M such that for any g-granular supermartingale T one can find a sequence on which M succeeds while T could not successfully save.

- Let M be the martingale that always bet one granule on side '1' if possible.
- For any martingale N, roughly, let X be the sequence such that $\frac{N(X\upharpoonright n)}{M(X\upharpoonright n)}$ is non-increasing.
- Then along with X the savings of N will be constrained, while M will succeed.

Granularity and saving strategies

	fine granularity	coarse granularity
"win by saving" for a single strategy	✓	×
"win by saving" for a class of countable strategies	✓	?

Timid betting strategies

Note that in the previous proof the martingales we constructed have the property that its wagers are linearly bounded by the granules.

Definition (timid martingale)

Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a nondecreasing function.

- A function $f\colon 2^{<\omega} \mapsto \mathbb{Q}$ is g-timid if it is g-granular and there is a constant c such that $f(\sigma) \leq c \cdot 2^{-g(|\sigma|)}$ for all $\sigma \in 2^{<\omega}$.
- A (super)martingale M is g-timid if its wager w_M is a g-timid function.

Timidness and saving strategies

Theorem (savings paradox for coarse timid strategies, [Barmpalias, Fang])

Let g be of coarse granularity. There exists a g-timid martingale M such that for any countable class C of g-timid supermartingales one can always find a sequence on which M succeeds while none of C could successfully save.

Theorem (weakness of timid martingales, [Barmpalias, Fang])

Let g be any order. From any g-timid supermartingale M one can construct a class C of countable g-granular supermartingales such that C successfully saves on any sequence M succeeds on.

Thank you for your attention!