On the convergence rate of d.c.e. approximations

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left-c.e. reals and Ω

Definition

Let $\{\alpha_s\}_{s\in\omega}$ be a computable nondecreasing sequence of rationals converging to α. We say that α is a left-c.e. real and $\{\alpha_s\}_{s\in\omega}$ is a left-c.e. approximation of α.

We define right-c.e. reals and approximations similarly. A real is computable if and only if it is both left-c.e. and right-c.e.

Fix an universal prefix-free machine U, its halting probability is defined as

$$\Omega = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|}.$$

Let

$$\Omega_s = \sum_{|\sigma|<s \& U[s](\sigma)\downarrow} 2^{-|\sigma|},$$

then $\{\Omega_s\}_{s\in\omega}$ is a left-c.e. approximation of Ω. Moreover, Chaitin proved that Ω is 1-random.

left-c.e. reals and Ω

Definition

A left-c.e. real α is *Solovay reducible* to a left-c.e. real β ($\alpha \leq_S \beta$) if there are a left-c.e. approximation $\{\alpha_s\}_{s\in\omega}$ of α, a left-c.e. approximation $\{\beta_s\}_{s\in\omega}$ of β and a constant c such that $\alpha - \alpha_s \leq c(\beta - \beta_s)$ for all $s \in \omega$.

A left-c.e. real is *Solovay complete* if every left-c.e. real is Solovay reducible to it.

With the works by Chaitin (1975), Solovay (1975), Calude, Hertling, Khoussainov, and Wang (2001), and Kucera and Slaman (2001), it has been shown that the following are equivalent.

1. α is a random left-c.e. real;
2. α is the halting probability of a universal prefix-free machine;
3. α is Solovay complete.

d.c.e. reals

Definition

A real is d.c.e. if it is the difference of two left-c.e. reals.

Proposition (Ambos-Spies, Weihrauch, and Zheng 2000)

A real α is d.c.e. if and only if there is a computable sequence of rationals $\{\alpha_s\}_{s\in\omega}$ converging to α such that the variation $\sum_{s\in\omega}|\alpha_{s+1}-\alpha_s|$ is bounded.

Random d.c.e. reals have the following nice property.

Proposition (Rettinger and Zheng 2005)

Random d.c.e. reals are either left-c.e. reals or right c.e. reals.

a derivation of the d.c.e. reals

Fix a left-c.e. approximation $\{\Omega_s\}_{s\in\omega}$ of Ω.

Definition

For a d.c.e. real α with d.c.e. approximations $\{\alpha_s\}_{s\in\omega}$, let

$$\partial\alpha = \partial\{\alpha_s\} = \lim_{s\to\infty} \frac{\alpha - \alpha_s}{\Omega - \Omega_s}.$$

Theorem (Barmpalias and Lewis-Pye 2017, refined by Miller 2017)

Let α be a d.c.e. real.

1. $\partial\alpha$ *converges and is independent of the choice of* $\{\alpha_s\}_{s\in\omega}$.
2. $\partial\alpha = 0$ *if and only if* α *is not random.*
3. $\partial\alpha > 0$ *if and only if* α *is random and left-c.e.*
4. $\partial\alpha < 0$ *if and only if* α *is random and right-c.e.*

a characterization of random d.c.e. reals

Theorem 1

Let γ be a d.c.e. real. If γ is random, then for any two d.c.e. approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of γ,

$$\lim_{s\to\infty} \frac{\gamma - b_s}{\gamma - a_s} = 1.$$

Proof.

As γ be a d.c.e. random real, then $\partial\{a_s\} = \partial\{b_s\} = \partial\gamma \neq 0$, i.e. ,

$$\lim_{s\to\infty} \frac{\gamma - a_s}{\Omega - \Omega_s} = \lim_{s\to\infty} \frac{\gamma - b_s}{\Omega - \Omega_s} \neq 0.$$

Then

$$\lim_{s\to\infty} \frac{\gamma - b_s}{\gamma - a_s} = \lim_{s\to\infty} \frac{\frac{\gamma - b_s}{\Omega - \Omega_s}}{\frac{\gamma - a_s}{\Omega - \Omega_s}} = 1.$$

□

a characterization of random d.c.e. reals

Theorem 2

Let γ be a Δ^0_2 real. If γ is not random, then for any $c > 1$ there exist two computable approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of γ such that

$$\liminf_{s\to\infty}\left|\frac{\gamma-b_s}{\gamma-a_s}\right| < \frac{1}{c} < c < \limsup_{s\to\infty}\left|\frac{\gamma-b_s}{\gamma-a_s}\right|. \tag{1}$$

Moreover, if γ is d.c.e. , $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ can also be chosen as d.c.e. approximations of γ.

Proof.

Let $\{r_s\}_{s\in\omega}$ be a computable approximation of γ. As γ is not random, let $T = \{I_i\}_{i\in\omega}$ be a Solovay test of intervals which covers γ. Given any $c > 0$, we construct two approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of γ as follows.

- Before stage 0, we say all interval I_i are *unused*.
- At stage $s \geq 0$, check if $r_s \in I_i$ for some unused interval I_i with $i \leq s$.
- If so, mark the least such interval I_i *used* and let $a_{2s} = b_{2s+1} = r_s$, $b_{2s} = a_{2s+1} = r_s - (c+1)\cdot\mu(I_i)$.
- Otherwise, let $a_{2s} = a_{2s+1} = b_{2s} = b_{2s+1} = r_s$.

Proof (Cont.)

1. $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ are computable approximations of γ.
   For $i \in \{0,1\}$, $|a_{2s+i} - r_s| \le (c+1)\cdot\mu(I_i) \to 0$. As $r_s \to \gamma$, so as $\{a_s\}_{s\in\omega}$. The same applies for $\{b_s\}_{s\in\omega}$.

2. If $\{r_s\}_{s\in\omega}$ is a d.c.e. approximation of γ, so are $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$.
   By the construction, the total variation of $\{a_s\}_{s\in\omega}$ ($\{b_s\}_{s\in\omega}$ as well) is bounded by the variation of $\{r_s\}_{s\in\omega}$ plus $\sum_{i\in\omega} 2(c+1)\mu(I_i)$, which is finite.

3. If $\gamma \in I_i$, I_i will eventually be used at some stage s, and $\left|\frac{\gamma-b_{2s}}{\gamma-a_{2s}}\right| = \left|\frac{\gamma-a_{2s+1}}{\gamma-b_{2s+1}}\right| > c$.
   For such I_i, $\mu(I_i) > |\gamma - r_s|$. Then

$$\left|\frac{\gamma - b_{2s}}{\gamma - a_{2s}}\right| = \left|\frac{\gamma - a_{2s+1}}{\gamma - b_{2s+1}}\right| = \left|1 + \frac{(c+1)\cdot\mu(I_i)}{\gamma - r_s}\right| \ge \frac{(c+1)\cdot\mu(I_i)}{|\gamma - r_s|} - 1 > c.$$

As there are infinitely many i such that $\gamma \in I_i$, we get (1) verified. □

a characterization of random d.c.e. reals

Corollary 3

For a d.c.e. real γ, the following are equivalent:

1. *γ is random;*
2. *for any two d.c.e. approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of γ, $\lim_{s\to\infty}\frac{\gamma-b_s}{\gamma-a_s}$ exists.*
3. *for any two d.c.e. approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of γ, $\lim_{s\to\infty}\frac{\gamma-b_s}{\gamma-a_s}=1$.*

an analog for random left-c.e. reals

Clearly, we also have:

If γ is a random left-c.e. real, then for any two left-c.e. approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of γ, $\lim_{s\to\infty} \frac{\gamma-b_s}{\gamma-a_s} = 1$.

How about the converse? Is the following statement true?

If γ is a nonrandom left-c.e. real, then there exists two left-c.e. approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of γ, such that $\lim_{s\to\infty} \frac{\gamma-b_s}{\gamma-a_s}$ doesn't exist.

an analog for random left-c.e. reals?

Corollary 4

There exists a set X which is not low for Ω such that for any two X-left-c.e. approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of Ω,

$$\lim_{s\to\infty}\frac{\Omega-b_s}{\Omega-a_s}=1.$$

Proof.

- Let X be a non-computable hyperimmune-free set. By a result by Miller and Nies, X is computably dominated but not low for Ω.
- Given an X-left-c.e. approximation $\{a_s\}_{s\in\omega}$, define two X-computable functions g and h:

$$g(n)=\max\{i\colon \Omega_i\le a_n\}\quad \text{and}\quad h(n)=\min\{i\colon \Omega_i\ge a_n\}$$

- As X is computably dominated, there exists two computable functions g' and h' dominate g^{-1} and h respectively, i.e. , $\forall^\infty n[g^{-1}(n)\le g'(n)]$ and $\forall^\infty n[h(n)\le h'(n)]$. Then $\forall^\infty n[\Omega_{g'^{-1}(n)}\le\Omega_{g(n)}\le a_n\le\Omega_{h(n)}\le\Omega_{h'(n)}]$. Thus, $\limsup_{s\to\infty}\frac{\Omega-a_s}{\Omega-\Omega_s}\le$ $\limsup_{s\to\infty}\frac{\Omega-\Omega_{g'^{-1}(s)}}{\Omega-\Omega_s}=1=\liminf_{s\to\infty}\frac{\Omega-\Omega_{h'(s)}}{\Omega-\Omega_s}\le\liminf_{s\to\infty}\frac{\Omega-a_s}{\Omega-\Omega_s}$.
- Similarly, we get $\lim_{s\to\infty}\frac{\Omega-b_s}{\Omega-\Omega_s}=1$. Thus, $\lim_{s\to\infty}\frac{\Omega-b_s}{\Omega-a_s}=1$.

$\square$

an analog for random left-c.e. reals?

Suppose the following is true.

If γ is a nonrandom left-c.e. real, then there exists two left-c.e. approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of γ, such that $\lim_{s\to\infty} \frac{\gamma-b_s}{\gamma-a_s}$ doesn't exist.

Then very likely by relativization we would get the following statement:

If X is not low for Ω, then there exist two X-left-c.e. approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of Ω such that $\lim_{s\to\infty} \frac{\gamma-b_s}{\gamma-a_s}$ doesn't exist.

This contradicts to Corollary 4. So we ask the following question.

Question:

Which randomness is sufficient to ensure that for any two left-c.e. approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of such a random real γ, $\lim_{s\to\infty} \frac{\gamma-b_s}{\gamma-a_s} = 1$?

speedup functions for approximating Ω

Theorem 5

For a strictly increasing function $f : \mathbb{N} \mapsto \mathbb{N}$, if f is dominated by a function g which is computable from some low for Ω set X, then

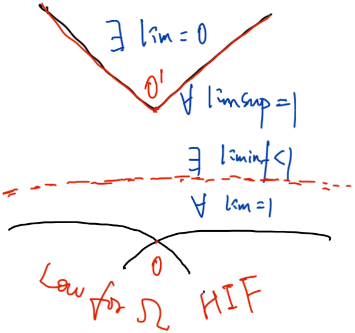
$$\lim_{s\to\infty} \frac{\Omega - \Omega_{f(s)}}{\Omega - \Omega_s} = 1.$$

Proof.

- Given a low for Ω set X and a function g computable from X, $\{\Omega_s\}_{s\in\omega}$ and $\{\Omega_{g(s)}\}_{s\in\omega}$ are X-left-c.e. approximations of Ω.
- As Ω is X-random, by relativizing Theorem 1, we get that $\lim_{s\to\infty} \frac{\Omega-\Omega_{g(s)}}{\Omega-\Omega_s} = 1$.
- As f is dominated by g, then $\liminf_{s\to\infty} \frac{\Omega-\Omega_{f(s)}}{\Omega-\Omega_s} \geq \liminf_{s\to\infty} \frac{\Omega-\Omega_{g(s)}}{\Omega-\Omega_s} = 1$.
- On the other hand, as f is strictly increasing, we have $\Omega_{f(s)} \geq \Omega_s$. Then $\limsup_{s\to\infty} \frac{\Omega-\Omega_{f(s)}}{\Omega-\Omega_s} \leq \limsup_{s\to\infty} \frac{\Omega-\Omega_s}{\Omega-\Omega_s} = 1$.

$\square$

speedup functions for approximating Ω



- As $\Omega \equiv_T 0'$, any Turing degree $\geq_T 0'$ contains a function f such that $\lim_{s\to\infty} \frac{\Omega - \Omega_{f(s)}}{\Omega - \Omega_s} = 0$.
- We can also prove that, for any strictly increasing function $f \not\geq_T 0'$, $\limsup_{s\to\infty} \frac{\Omega - \Omega_{f(s)}}{\Omega - \Omega_s} = 1$.
- Our corollary above tells that if a strictly increasing function f is dominated by a low for Ω function, then $\lim_{s\to\infty} \frac{\Omega - \Omega_{f(s)}}{\Omega - \Omega_s} = 1$.
- However, at the moment we still don't know which degrees exactly contain an increasing function f such that $\liminf_{s\to\infty} \frac{\Omega - \Omega_{f(s)}}{\Omega - \Omega_s} < 1$.

speedup functions for approximating Ω

Theorem 6

Let f be a strictly increasing function. If $\limsup_{s\to\infty} \frac{\Omega-\Omega_{f(s)}}{\Omega-\Omega_s} < 1$ *then* $f \geq_T \Omega \equiv_T 0'$.

Proof.

Given f as stated, there exists $\epsilon > 0$ and s_0 such that for all $s \geq s_0$,

$$\frac{\Omega - \Omega_{f(s)}}{\Omega - \Omega_s} < 1 - \epsilon \implies \Omega - \Omega_s < (\Omega_{f(s)} - \Omega_s)/\epsilon.$$

We compute Ω from f as follows. For each n, find some σ and $s \geq s_0$ such that $|\sigma| > n$, $\sigma^\frown 00 \prec \Omega_s$ and $\Omega_{f(s)} - \Omega_s < \epsilon \cdot 2^{-|\sigma|-2}$. We claim $\Omega \upharpoonright n = \sigma \upharpoonright n$.

- Since Ω is random, it has infinitely many blocks of 00. On the other hand, as $\Omega_s \to \Omega$, $(\Omega_{f(s)} - \Omega_s) \leq (\Omega_{f(s)} - \Omega_s) \to 0$. Thus, the desired σ and s exist.
- By the choice of s, we have $\Omega - \Omega_s < (\Omega_{f(s)} - \Omega_s)/\epsilon < 2^{-|\sigma|-2}$. Combined with $\sigma^\frown 00 \prec \Omega_s$, we have $\sigma \prec \Omega$.

Thus, $0' \equiv_T \Omega \leq_T f$. □

Thank you for your attention!