Convergence rates of computably approximable reals

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left-c.e. reals and Ω

Definition

Let $\{\alpha_s\}_{s\in\omega}$ be a computable nondecreasing sequence of rationals converging to α. We say that α is a left-c.e. real and $\{\alpha_s\}_{s\in\omega}$ is a left-c.e. approximation of α.

We define right-c.e. reals and approximations similarly. A real is computable if and only if it is both left-c.e. and right-c.e.

Fix an universal prefix-free machine U, its halting probability is defined as

$$\Omega = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|}.$$

Let

$$\Omega_s = \sum_{|\sigma|<s \& U[s](\sigma)\downarrow} 2^{-|\sigma|},$$

then $\{\Omega_s\}_{s\in\omega}$ is a left-c.e. approximation of Ω. Moreover, Ω is Martin-Löf random.

d.c.e. reals

Definition

A real is d.c.e. if it is the difference of two left-c.e. reals.

Proposition (Ambos-Spies, Weihrauch, and Zheng 2000)

A real α is d.c.e. if and only if there is a computable sequence of rationals $\{\alpha_s\}_{s\in\omega}$ converging to α such that the variation $\sum_{s\in\omega}|\alpha_{s+1}-\alpha_s|$ is bounded.

Random d.c.e. reals have the following nice property.

Proposition (Rettinger and Zheng 2005)

Random d.c.e. reals are either left-c.e. reals or right c.e. reals.

a derivation of the d.c.e. reals

Fix a left-c.e. approximation $\{\Omega_s\}_{s\in\omega}$ of Ω.

Definition

For a d.c.e. real α with d.c.e. approximations $\{\alpha_s\}_{s\in\omega}$, let

$$\partial\alpha = \partial\{\alpha_s\} = \lim_{s\to\infty} \frac{\alpha - \alpha_s}{\Omega - \Omega_s}.$$

Theorem (Barmpalias and Lewis-Pye 2017, refined by Miller 2017)

Let α be a d.c.e. real.

1. *$\partial\alpha$ converges and is independent of the choice of $\{\alpha_s\}_{s\in\omega}$.*
2. *$\partial\alpha = 0$ if and only if α is not random.*
3. *$\partial\alpha > 0$ if and only if α is random and left-c.e.*
4. *$\partial\alpha < 0$ if and only if α is random and right-c.e.*

a characterization of random d.c.e. reals

Theorem 1

Let γ be a d.c.e. real. If γ is random, then for any two d.c.e. approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of γ,

$$\lim_{s\to\infty} \frac{\gamma - b_s}{\gamma - a_s} = 1.$$

Proof.

As γ be a d.c.e. random real, then $\partial\{a_s\} = \partial\{b_s\} = \partial\gamma \neq 0$, i.e. ,

$$\lim_{s\to\infty} \frac{\gamma - a_s}{\Omega - \Omega_s} = \lim_{s\to\infty} \frac{\gamma - b_s}{\Omega - \Omega_s} \neq 0.$$

Then

$$\lim_{s\to\infty} \frac{\gamma - b_s}{\gamma - a_s} = \lim_{s\to\infty} \frac{\frac{\gamma - b_s}{\Omega - \Omega_s}}{\frac{\gamma - a_s}{\Omega - \Omega_s}} = 1.$$

□

a characterization of random d.c.e. reals

Theorem 2

Let γ be a nonrandom irrational real. If γ has a computable approximation $\{r_s\}_{s\in\omega}$, then it has another computable approximation $\{a_s\}_{s\in\omega}$ such that

$$\limsup_{s\to\infty}\left|\frac{\gamma - a_s}{\gamma - r_s}\right| = +\infty. \tag{1}$$

Moreover, if $\{r_s\}_{s\in\omega}$ is d.c.e. , $\{a_s\}_{s\in\omega}$ can also be made d.c.e. .

Proof.

As γ is not random, let $\{U_i\}_{i\in\omega}$ be a Martin-Löf test which covers γ. We construct an approximation $\{a_s\}_{s\in\omega}$ of γ such that $\forall n\,\exists s \geq n \left[\left|\frac{\gamma-a_s}{\gamma-r_s}\right| \geq 2^n - 1\right]$, as well as a series of strings $\{\sigma_i\}_{i\in\omega}$ such that $\sigma_i \in U_{2i}$ and $\sigma_i \prec \gamma$.

At then beginning, we let $\sigma_i \uparrow$ for all $i \in \omega$. At each stage $s \geq 0$, we find the least integer n such that $\sigma_n \uparrow$. Then check if there exists an index $j < n$ such that $\sigma_j \not\prec r_s$ and do as follows.

- If exists, let j be the least such index and let $\sigma_i \uparrow$ for all $i \geq j$. Then let $a_s = r_s$ and go to stage $s + 1$.
- Otherwise, search for the least $t \geq s$ such that there exists $\sigma \in U_{2n}$ with $\sigma \prec r_t$. Then let $\sigma_n = \sigma$, $a_i = r_i$ for all $s \leq i < t$, $a_t = r_t - 2^{n-|\sigma|}$, and go to stage $t + 1$.

a characterization of random d.c.e. reals

Proof (Cont.)

1. For each $n \in \omega$, σ_n is defined and is the unique prefix of γ in U_{2n}.
   As $\gamma \in [\![U_{2n}]\!]$ and U_{2n} is prefix-free , there exists a unique $\sigma \in U_{2n}$ such that $\sigma \prec \gamma$. σ_n will eventually be defined as σ.

2. $\{a_s\}_{s\in\omega}$ is a computable approximation of γ. If $\{r_s\}_{s\in\omega}$ is d.c.e. , $\{a_s\}_{s\in\omega}$ is d.c.e. as well.
   In the construction, we have $a_s = r_s$ or $a_s = r_s - 2^{n_s - |\sigma_{n_s,s}|}$. In the latter case, n_s is the least integer n such that $\sigma_n \uparrow$ at stage s. Then
   $\lim_{s\to\infty}|a_s - r_s| \leq \lim_{s\to\infty} 2^{n_s-|\sigma_{n_s,s}|} \leq \lim_{s\to\infty} 2^{n_s} \cdot 2^{-2n_s} = \lim_{s\to\infty} 2^{-n_s} = 0$. Thus, $\lim_{s\to\infty} a_s = \gamma$.
   Moreover, the variation of $\{a_s\}_{s\in\omega}$ is bounded by the variation of $\{r_s\}_{s\in\omega}$ plus the sum of $2 \cdot 2^{n_s-|\sigma_{n_s,s}|}$ for all stage s where σ_{n_s} is newly defined or redefined. As we only define σ_n to be some member in U_{2n}, that sum is then bounded by
   $\sum_{n\in\omega} 2 \cdot 2^n \mu(U_{2n}) \leq \sum_{n\in\omega} 2^{1-n} = 4$. Thus, if $\{r_s\}_{s\in\omega}$ has finite variation, so does $\{a_s\}_{s\in\omega}$.

3. If σ_n is lastly defined at stage s, then $\left|\frac{\gamma-a_s}{\gamma-r_s}\right| \geq 2^n - 1$.
   As $\sigma_n \prec r_s$ and $\sigma_n \prec \gamma$. we have $0 < |\gamma - r_s| \leq 2^{-|\sigma_n|}$, and
   $|\gamma - a_s| = |\gamma - r_s + 2^{n-|\sigma_n|}| \geq 2^{n-|\sigma_n|} - |\gamma - r_s| \geq (2^n - 1) \cdot |\gamma - r_s|$.

□

a characterization of random d.c.e. reals

Corollary 3

For a d.c.e. real γ, the following are equivalent:

1. *γ is random;*
2. *for any two d.c.e. approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of γ, $\lim_{s\to\infty} \frac{\gamma-b_s}{\gamma-a_s}$ exists.*
3. *for any two d.c.e. approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of γ, $\lim_{s\to\infty} \frac{\gamma-b_s}{\gamma-a_s} = 1$.*

an analog for left-c.e. reals

Definition

Let $\{a_i\}_{i\in\omega}$ be a series of computable nondecreasing converging rationals and c be an positive integer. We say that $\{(a_i, a_i + c(a_{i+1} - a_i))\}_{i\in\omega}$ is an *approximating Solovay test*. A left-c.e. real α is *approximately random* if it passes all approximating Solovay test, i.e. , $\alpha \notin (a_i, a_i + c(a_{i+1} - a_i))$ for all but finitely many i.

We can prove the following analog theorem for left-c.e. reals.

Theorem 4

For a left-c.e. real γ, the following are equivalent:

1. *γ is approximately random;*
2. *for any left-c.e. approximation $\{a_s\}_{s\in\omega}$ of γ,* $\liminf_{s\to\infty} \frac{\gamma - a_{s+1}}{\gamma - a_s} = 1$;
3. *for any two left-c.e. approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of γ,* $\lim_{s\to\infty} \frac{\gamma - b_s}{\gamma - a_s}$ *exists;*
4. *for any two left-c.e. approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of γ,* $\lim_{s\to\infty} \frac{\gamma - b_s}{\gamma - a_s} = 1$.

low for approximately random

Definition

Let AR be the set of all approximately random left-c.e. reals, and for a set X, let AR^X be the set of all approximately random left-c.e. reals relative to X. X is *low for approximately random* if $AR = AR^X$.

Theorem 5

If A is HIF, then A is low for approximately random.

Theorem 6

If A is low for Ω*, then A is low for* Ω *to be approximately random.*

speedup functions for approximating Ω

Theorem 7

Let f be a strictly increasing function. If $\limsup_{s\to\infty} \frac{\Omega-\Omega_{f(s)}}{\Omega-\Omega_s} < 1$ *then* $f \geq_T \Omega \equiv_T 0'$.

Proof.

Given f as stated, there exists $\epsilon > 0$ and s_0 such that for all $s \geq s_0$,

$$\frac{\Omega - \Omega_{f(s)}}{\Omega - \Omega_s} < 1 - \epsilon \implies \Omega - \Omega_s < \left(\Omega_{f(s)} - \Omega_s\right)/\epsilon.$$

We compute Ω from f as follows. For each n, find some σ and $s \geq s_0$ such that $|\sigma| > n$, $\sigma^\frown 00 \prec \Omega_s$ and $\Omega_{f(s)} - \Omega_s < \epsilon \cdot 2^{-|\sigma|-2}$. We claim $\Omega \upharpoonright n = \sigma \upharpoonright n$.

- Since Ω is random, it has infinitely many blocks of 00. On the other hand, as $\Omega_s \to \Omega$, $\left(\Omega_{f(s)} - \Omega_s\right) \leq \left(\Omega_{f(s)} - \Omega_s\right) \to 0$. Thus, the desired σ and s exist.
- By the choice of s, we have $\Omega - \Omega_s < \left(\Omega_{f(s)} - \Omega_s\right)/\epsilon < 2^{-|\sigma|-2}$. Combined with $\sigma^\frown 00 \prec \Omega_s$, we have $\sigma \prec \Omega$.

Thus, $0' \equiv_T \Omega \leq_T f$. $\square$

Thank you for your attention!