

Integer-valued martingales and cl-reductions

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Integer-valued martingales

Definition 1 (integer-valued martingale)

An integer-valued martingale is a function $f: 2^{<\omega} \rightarrow \mathbb{N}$ such that $f(\sigma \hat{\ } 0) + f(\sigma \hat{\ } 1) = 2f(\sigma)$ holds for all $\sigma \in 2^{<\omega}$.

- ▶ A left-c.e. integer-valued martingale is always computable.
- ▶ A computable integer-valued supermartingale is bounded above by a computable integer-valued martingale.

Definition 2 (integer-valued randomness)

An infinite binary sequence X is integer-valued random (IVR) if there is no computable integer-valued (super)martingale succeeds on it.

Definition 3

A Turing reduction Φ is d -cl for some integer $d \geq 0$, if for all $\sigma \in 2^{\geq d+1}$, either $\Phi^\sigma(|\sigma| - d - 1) \downarrow$ or $\Phi^\rho(n) \uparrow$ for all $\rho \geq \sigma$ and $n \geq |\sigma| - d - 1$. It is d -cl-tt if only the first cause applies. It is cl (cl-tt) if it is d -cl (d -cl-tt) for some integer d .

Give a d -cl Turing reduction Φ , define a partial map $T: 2^{\geq d} \mapsto 2^{\geq 0}$ as follows:

- ▶ $T(\sigma) = \lambda$ for all $|\sigma| = d$;
- ▶ $T(\sigma) = \tau$ if $\Phi^\sigma \upharpoonright (|\sigma| - d) = \tau$.

Then T is a partial map $T: 2^{\geq d} \mapsto 2^{\geq 0}$ with the following property:

$$\text{If } T(\sigma) = \tau \text{ then } |\sigma| = |\tau| + d \text{ and } T(\eta) \downarrow < T(\sigma) \text{ for all } \eta < \sigma. \quad (1)$$

Clearly, any partial map $T: 2^{\geq d} \mapsto 2^{\geq 0}$ with property (1) can also be converted to a cl reduction.

- ▶ $|T(\sigma)|$ is a left-c.e. integer-valued supermartingale.
- ▶ In case Φ is cl-tt, $|T(\sigma)|$ is a computable integer-valued martingale.

correspondence between integer-valued martingales and cl-tt reductions

Lemma 4

For any integer-valued computable martingale f , there is a cl-tt tree map T and a natural number c such that $|T^{-1}(\sigma)| = f(\sigma) + c$ for all $\sigma \in 2^{<\omega}$. Moreover, for any X , we have $\lim_n f(X \upharpoonright n) = \infty$ if and only if $|T^{-1}(X)| = \infty$.

Theorem 5

X is IVR if and only if for every cl-tt reduction Φ , there are only finitely many oracles compute X via Φ .

proof idea of Lemma 4

Lemma 4

For any integer-valued computable martingale f , there is a cl-tt tree map T and a natural number c such that $|T^{-1}(\sigma)| = f(\sigma) + c$ for all $\sigma \in 2^{<\omega}$. Moreover, for any X , we have $\lim_n f(X \upharpoonright n) = \infty$ if and only if $|T^{-1}(X)| = \infty$.

- ▶ Let d be the least integer such that $f(\lambda) \leq 2^d$.
- ▶ Let $c = 2^d - f(\lambda)$ and $g(\sigma) = f(\sigma) + c$ for all $\sigma \in 2^{<\omega}$.
- ▶ Then g is an integer-valued computable martingale with $g(\lambda) = 2^d$.

proof idea of Lemma 4

We define a tree map $T: 2^{\geq d} \rightarrow 2^{\geq 0}$ such that the following properties are satisfied for all strings $\sigma \in 2^{<\omega}$.

- ❶ $|T^{-1}(\sigma)| = g(\sigma)$;
- ❷ if $\sigma_0 < \sigma$, and $g(\rho) \geq g(\sigma_0)$ for all $\sigma_0 < \rho \leq \sigma$, then every string in $T^{-1}(\sigma_0)$ has an extension in $T^{-1}(\sigma)$.

construction:

At stage 0, let $T(\tau) = \lambda$ for all $\tau \in 2^d$.

At stage $s + 1$, for each $\sigma \in 2^s$ we do the following:

- ❶ Find $i \in \{0, 1\}$ such that $g(\sigma^\frown i) \leq g(\sigma^\frown (1 - i))$.
- ❷ Find $V \subset T^{-1}(\sigma)$ such that $|V| = g(\sigma^\frown i)$ and the following property holds: if σ_0 is the longest prefix of σ such that $g(\sigma_0) \leq g(\sigma^\frown i)$, then each string in $T^{-1}(\sigma_0)$ has a unique extension in V .
- ❸ Let $T(\tau^\frown 0) = \sigma^\frown i$ and $T(\tau^\frown 1) = \sigma^\frown (1 - i)$ for each $\tau \in V$, and $T(\tau^\frown 0) = (\tau^\frown 1) = \sigma^\frown (1 - i)$ for each $\tau \in T^{-1}(\sigma) \setminus V$.

A simple property of integer-valued martingales

Easy fact:

For any integer martingale f , any string σ_0 , there exist a string $\sigma \geq \sigma_0$ and a constant $r \leq f(\sigma_0)$ such that $f(\tau) = r$ for all $\tau \geq \sigma$.

Proof.

Let σ be the least string in $L = \{\tau : \tau \geq \sigma_0\}$ such that $f(\sigma) = \min_{\tau \in L} f(\tau)$. Let $r = f(\sigma)$.

- ▶ Given $\eta \geq \sigma$, suppose $f(\eta \hat{1}) = r + t$ and $f(\tau) = r$ for all $\eta \geq \tau \geq \sigma$.
- ▶ If $t < 0$, then $f(\eta \hat{1}) = r + t < r = \min_{\tau \in L} f(\tau)$. $\nmid$
- ▶ If $t > 0$, then $f(\eta \hat{0}) = r - t < r = \min_{\tau \in L} f(\tau)$. $\nmid$
- ▶ Thus, $t = 0$ and $f(\eta \hat{1}) = f(\eta \hat{0}) = r$.



a special integer-valued martingale

Let M be the integer martingale which starts with initial capital 1, and always bets 1 on '1' if possible. That is,

- 1 $M(\lambda) = 1$.
- 2 For any $\sigma \in 2^{<\omega}$ and $i \in \{0, 1\}$,

$$M(\sigma \hat{\ } i) = \begin{cases} M(\sigma) + 1 & \text{if } M(\sigma) \geq 1 \text{ \& } i = 1, \\ M(\sigma) - 1 & \text{if } M(\sigma) \geq 1 \text{ \& } i = 0, \\ M(\sigma) & \text{if } M(\sigma) = 0. \end{cases}$$

properties of integer-valued martingales

For $c < M(\sigma)$, let $\llbracket \sigma \rrbracket^{M>c} = \{\tau \geq \sigma : \forall \sigma \leq \eta \leq \tau, M(\eta) > c\}$.

Lemma 6

For any integer martingale f and any string σ_0 with $f(\sigma_0) < M(\sigma_0)$, there exist a string $\sigma \geq \sigma_0$ and a constant $r < M(\sigma_0)$ such that $f(\tau) = r$ for all $\tau \in \llbracket \sigma \rrbracket^{M>r}$.

Proof.

Let σ be the least string in $L = \{\tau \geq \sigma_0 : f(\tau) < M(\tau)\}$ such that $f(\sigma) = \min_{\tau \in L} f(\tau)$. Let $r = f(\sigma)$.

- ▶ Given $\eta \in \llbracket \sigma \rrbracket^{M>r}$, suppose $f(\eta \hat{\ } 1) = r + t$ and $f(\tau) = r$ for all $\eta \geq \tau \in \llbracket \sigma \rrbracket^{M>r}$.
- ▶ If $t < 0$, then $f(\eta \hat{\ } 1) < r < m(\eta) < M(\eta \hat{\ } 1)$ and $f(\eta \hat{\ } 1) < r = \min_{\tau \in L} f(\tau)$. $\nmid$
- ▶ If $t > 0$, then $f(\eta \hat{\ } 0) = r - t \leq r - 1 < m(\eta) - 1 = M(\eta \hat{\ } 0)$ and $f(\eta \hat{\ } 0) < r = \min_{\tau \in L} f(\tau)$. $\nmid$
- ▶ Thus, $t = 0$ and $f(\eta \hat{\ } 1) = f(\eta \hat{\ } 0) = r$.



properties of integer-valued martingales

Lemma 7

For any integer martingale f and any string σ_0 with $f(\sigma_0) = q_0M(\sigma_0) + r_0$ and $0 \leq r_0 < M(\sigma_0)$, there exist a string $\sigma \geq \sigma_0$ and constants $q \leq q_0, r < M(\sigma)$ such that $f(\tau) = qM(\tau) + r$ for all $\tau \in \llbracket \sigma \rrbracket^{M>r}$.

Proof.

- ▶ For all σ in $L = \{\tau \geq \sigma_0 : M(\tau) > 0\}$, let q_σ, r_σ be natural numbers such that $f(\sigma) = q_\sigma M(\sigma) + r_\sigma$ and $0 \leq r_\sigma < M(\sigma)$.
- ▶ Define an order on the pairs (q_σ, r_σ) : $(q_1, r_1) < (q_2, r_2)$ iff $q_1 < q_2$ or $q_1 = q_2 \& r_1 < r_2$.
- ▶ Let σ be the least string in L such that (q_σ, r_σ) is minimal. Let $q = q_\sigma$ and $r = r_\sigma$.

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- ▶ Given $\eta \in \llbracket \sigma \rrbracket^{M>r}$, suppose $f(\eta \hat{1}) = f(\eta) + t$ and $f(\tau) = qM(\tau) + r$ for all $\eta \geq \tau \in \llbracket \sigma \rrbracket^{M>r}$.
 - ▶ If $t < q$, then $f(\eta \hat{1}) = qM(\eta) + r + t < qM(\eta \hat{1}) + r$. $\nmid$
 - ▶ If $t > q$, then $f(\eta \hat{0}) = qM(\eta) + r - t \leq qM(\eta \hat{0}) + r - 1$ and $M(\eta \hat{0}) = M(\eta) - 1 > r - 1$. $\nmid$
 - ▶ Thus, $t = q, f(\eta \hat{1}) = qM(\eta \hat{1}) + r$ and $f(\eta \hat{0}) = qM(\eta \hat{0}) + r$.

IVR and cl-tt computability

Theorem 8

There exists a non-IVR sequence such that only countably many oracles cl-tt-compute it.

Proof idea:

- ▶ Given a single cl-tt map T , by Lemme 7, we can find σ_0, q, r such that $|T^{-1}(\tau)| = qM(\tau) + r$ for all $\tau \in \llbracket \sigma_0 \rrbracket^{M>r}$.
- ▶ Among any $q + 1$ prefix-free strings there is at least one string η such that $|T^{-1}(\sigma) \cap [\eta]| < M(\sigma) - r$ for some $\sigma \in \llbracket \sigma_0 \rrbracket^{M>r}$.
- ▶ By Lemma 6, we can find $\sigma \leq \sigma_1 \in \llbracket \sigma_0 \rrbracket^{M>r}$ and $r_1 < M(\sigma)$ such that $|T^{-1}(\tau) \cap [\eta]| = r_1$ for all $\tau \in \llbracket \sigma_1 \rrbracket^{M>r}$.

In this way, we can construct X and enumerate a tree S such that:

- ▶ $\lim_{n \rightarrow \infty} M(X \upharpoonright n) = \infty$.
- ▶ for all string $\eta \notin S$, $T^{-1}(X) \cap [\eta]$ is finite.
- ▶ S has at most q path, thus $T^{-1}(X)$ is countable.

Let $\{T_i\}_{i \in \omega}$ be a list of all cl-tt maps. Combining the above strategy for each T_i , we can construct a sequence X such that $\lim_{n \rightarrow \infty} M(X \upharpoonright n) = \infty$ and for each i , $T_i^{-1}(X)$ is countable.



Thank you for your attention!