Integer-valued martingales and cl-Turing reductions

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Integer-valued martingales

Definition 1 (integer-valued martingale)

An integer-valued supermartingale is a function $f\colon 2^{<\omega} \to \mathbb{N}$ such that for all $\sigma \in 2^{<\omega}$

$$f(\sigma^\frown 0) + f(\sigma^\frown 1) \leq 2f(\sigma).$$

f is an integer-valued martingale if only equality applies.

- f is a left-c.e. function if there are uniformly computable functions $\{f_s\}_{s\in\omega}$ such that $f(\sigma) = \lim_s f_s(\sigma)$ and $f_s(\sigma) \leq f_{s+1}(\sigma)$.
- A left-c.e. integer-valued martingale is always computable.
- A computable integer-valued supermartingale is bounded above by a computable integer-valued martingale.

Definition 2 (integer-valued randomness)

Given an infinite binary sequence X.

- A (super)martingale f succeeds on X if $\lim_{n\to\infty} f(X \upharpoonright n) = +\infty$.
- X is integer-valued random (IVR) if there is no computable integer-valued (super)martingale succeeds on it.

cl-Turing reductions

Definition 3

A Turing reduction Φ is d-cl for some integer $d \geq 0$, if for all $\sigma \in 2^{\geq d+1}$, either $\Phi^{\sigma}(|\sigma| - d - 1) \downarrow$ or $\Phi^{\rho}(n) \uparrow$ for all $\rho \succeq \sigma$ and $n \geq |\sigma| - d - 1$. It is d-cl-tt if only the first cause applies. It is cl (cl-tt) if it is d-cl (d-cl-tt) for some integer d.

Give a d-cl Turing reduction Φ, define a partial map $T\colon 2^{\geq d} \mapsto 2^{\geq 0}$ as follows:

- $T(\sigma) = \lambda$ for all $|\sigma| = d$;
- $T(\sigma) = \tau$ if $\Phi^{\sigma} \upharpoonright (|\sigma| - d) = \tau$.

Then T is a partial map $T\colon 2^{\geq d} \mapsto 2^{\geq 0}$ with the following property:

$$\text{If } T(\sigma) = \tau \text{ then } |\sigma| = |\tau| + d \text{ and } T(\eta) \downarrow \prec T(\sigma) \text{ for all } \eta \prec \sigma. \tag{1}$$

Clearly, any partial map $T\colon 2^{\geq d} \mapsto 2^{\geq 0}$ with property (1) can also be converted to a cl reduction.

- $|T^{-1}(\sigma)|$ is a left-c.e. integer-valued supermartingale.
- In case Φ is cl-tt, $|T^{-1}(\sigma)|$ is a computable integer-valued martingale.

correspondence between integer-valued martingales and cl-tt reductions

Lemma 4

For any integer-valued computable martingale f, there is a cl-tt tree map T and a natural number c such that $|T^{-1}(\sigma)| = f(\sigma) + c$ for all $\sigma \in 2^{<\omega}$. Moreover, for any X, we have $\lim_n f(X \upharpoonright n) = \infty$ if and only if $|T^{-1}(X)| = \infty$.

Theorem 5

X is IVR if and only if for every cl-tt reduction Φ, there are only finitely many oracles compute X via Φ.

proof idea of Lemma 4

Lemma 4

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- Let d be the least integer such that $f(\lambda) \leq 2^d$.
- Let $c = 2^d - f(\lambda)$ and $g(\sigma) = f(\sigma) + c$ for all $\sigma \in 2^{<\omega}$.
- Then g is an integer-valued computable martingale with $g(\lambda) = 2^d$.

proof idea of Lemma 4

We define a tree map $T\colon 2^{\geq d} \to 2^{\geq 0}$ such that the following properties are satisfied for all strings $\sigma \in 2^{<\omega}$.

1. $|T^{-1}(\sigma)| = g(\sigma)$;
2. if $\sigma_0 \prec \sigma$, and $g(\rho) \geq g(\sigma_0)$ for all $\sigma_0 \prec \rho \preceq \sigma$, then every string in $T^{-1}(\sigma_0)$ has an extension in $T^{-1}(\sigma)$.

construction:

At stage 0, let $T(\tau) = \lambda$ for all $\tau \in 2^d$.
At stage $s+1$, for each $\sigma \in 2^s$ we do the following:

1. Find $i \in \{0, 1\}$ such that $g(\sigma^\frown i) \leq g(\sigma^\frown(1-i))$.
2. Find $V \subset T^{-1}(\sigma)$ such that $|V| = g(\sigma^\frown i)$ and the following property holds: if σ_0 is the longest prefix of σ such that $g(\sigma_0) \leq g(\sigma^\frown i)$, then each string in $T^{-1}(\sigma_0)$ has a unique extension in V.
3. Let $T(\tau^\frown 0) = \sigma^\frown i$ and $T(\tau^\frown 1) = \sigma^\frown(1-i)$ for each $\tau \in V$, and
$T(\tau^\frown 0) = (\tau^\frown 1) = \sigma^\frown(1-i)$ for each $\tau \in T^{-1}(\sigma) \setminus V$.

A simple property of integer-valued martingales

Easy fact:

For any integer-valued supermartingale f, any string σ_0, there exist a string $\sigma \succeq \sigma_0$ and a constant $r \leq f(\sigma_0)$ such that $f(\tau) = r$ for all $\tau \succeq \sigma$.

Proof.

Let σ be the least string in $L = \{\tau \colon \tau \succeq \sigma_0\}$ such that $f(\sigma) = \min_{\tau \in L} f(\tau)$. Let $r = f(\sigma)$.

- Given $\eta \succeq \sigma$, suppose $f(\eta^\frown 1) = r + t$ and $f(\tau) = r$ for all $\eta \succeq \tau \succeq \sigma$.
- If $t < 0$, then $f(\eta^\frown 1) = r + t < r = \min_{\tau \in L} f(\tau)$. $\lightning$
- If $t > 0$, then $f(\eta^\frown 0) \leq r - t < r = \min_{\tau \in L} f(\tau)$. $\lightning$
- Thus, $t = 0$ and $f(\eta^\frown 1) = f(\eta^\frown 0) = r$.

$\square$

a special integer-valued martingale

Let M be the integer-valued martingale which starts with initial capital 1, and always bets 1 on '1' if possible. That is,

1. $M(\lambda) = 1$.
2. For any $\sigma \in 2^{<\omega}$ and $i \in \{0, 1\}$,

$$M(\sigma^\frown i) = \begin{cases} M(\sigma) + 1 & \text{if } M(\sigma) \geq 1 \ \& \ i = 1, \\ M(\sigma) - 1 & \text{if } M(\sigma) \geq 1 \ \& \ i = 0, \\ M(\sigma) & \text{if } M(\sigma) = 0. \end{cases}$$

For $c < M(\sigma)$, let $[\![\sigma]\!]^{M>c} = \{\tau \succeq \sigma \colon \forall \sigma \preceq \eta \preceq \tau\, M(\eta) > c\}$.

Lemma 6

For any integer-valued supermartingale f and any string σ_0 with $f(\sigma_0) < M(\sigma_0)$, there exist a string $\sigma \succeq \sigma_0$ and a constant $r < M(\sigma_0)$ such that $f(\tau) = r$ for all $\tau \in [\![\sigma]\!]^{M>r}$.

Proof.

Let σ be the least string in $L = \{\tau \succeq \sigma_0 \colon f(\tau) < M(\tau)\}$ such that $f(\sigma) = \min_{\tau \in L} f(\tau)$. Let $r = f(\sigma)$.

- Given $\eta \in [\![\sigma]\!]^{M>r}$, suppose $f(\eta^\frown 1) = r + t$ and $f(\tau) = r$ for all $\eta \succeq \tau \in [\![\sigma]\!]^{M>r}$.
- If $t < 0$, then $f(\eta^\frown 1) < r < m(\eta) < M(\eta^\frown 1)$ and $f(\eta^\frown 1) < r = \min_{\tau \in L} f(\tau)$. $\lightning$
- If $t > 0$, then $f(\eta^\frown 0) \le r - t \le r - 1 < m(\eta) - 1 = M(\eta^\frown 0)$ and $f(\eta^\frown 0) < r = \min_{\tau \in L} f(\tau)$. $\lightning$
- Thus, $t = 0$ and $f(\eta^\frown 1) = f(\eta^\frown 0) = r$.

$\square$

properties of integer-valued martingales

Lemma 7

For any integer-valued supermartingale f and any string σ_0 with $f(\sigma_0) = q_0 M(\sigma_0) + r_0$ and $0 \le r_0 < M(\sigma_0)$, there exist a string $\sigma \succeq \sigma_0$ and constants $q \le q_0, r < M(\sigma)$ such that $f(\tau) = qM(\tau) + r$ for all $\tau \in [\![\sigma]\!]^{M>r}$.

Proof.

- For all σ in $L = \{\tau \succeq \sigma_0 \colon M(\tau) > 0\}$, let q_σ, r_σ be natural numbers such that $f(\sigma) = q_\sigma M(\sigma) + r_\sigma$ and $0 \le r_\sigma < M(\sigma)$.
- Define an order on the pairs (q_σ, r_σ): $(q_1, r_1) < (q_2, r_2)$ iff $q_1 < q_2$ or $q_1 = q_2 \& r_1 < r_2$.
- Let σ be the least string in L such that (q_σ, r_σ) is minimal. Let $q = q_\sigma$ and $r = r_\sigma$.

- Given $\eta \in [\![\sigma]\!]^{M>r}$, suppose $f(\eta^\frown 1) = f(\eta) + t$ and $f(\tau) = qM(\tau) + r$ for all $\eta \succeq \tau \in [\![\sigma]\!]^{M>r}$.
- If $t < q$, then $f(\eta^\frown 1) = qM(\eta) + r + t < qM(\eta^\frown 1) + r$. $\lightning$
- If $t > q$, then $f(\eta^\frown 0) \le qM(\eta) + r - t \le qM(\eta^\frown 0) + r - 1$ and $M(\eta^\frown 0) = M(\eta) - 1 > r - 1$. $\lightning$
- Thus, $t = q, f(\eta^\frown 1) = qM(\eta^\frown 1) + r$ and $f(\eta^\frown 0) = qM(\eta^\frown 0) + r$.

$\square$

IVR and cl computability

Theorem 8

There exists a non-IVR sequence such that only countably many oracles cl-compute it.

Proof idea:

- Given a single cl-tt map T, by Lemme 7, we can find σ_0, q, r such that $|T^{-1}(\tau)| = qM(\tau) + r$ for all $\tau \in [\![\sigma_0]\!]^{M>r}$.
- Among any $q+1$ prefix-free strings there is at least one string η such that $|T^{-1}(\sigma) \cap [\eta]| < M(\sigma) - r$ for some $\sigma \in [\![\sigma_0]\!]^{M>r}$.
- By Lemma 6, we can find $\sigma \preceq \sigma_1 \in [\![\sigma_0]\!]^{M>r}$ and $r_1 < M(\sigma)$ such that $|T^{-1}(\tau) \cap [\eta]| = r_1$ for all $\tau \in [\![\sigma_1]\!]^{M>r}$.

In this way, we can construct X and enumerate a tree S such that:

- $\lim_{n\to\infty} M(X \upharpoonright n) = \infty$.
- for all string $\eta \notin S$, $T^{-1}(X) \cap [\eta]$ is finite.
- S has at most q path, thus $T^{-1}(X)$ is countable.

Let $\{T_i\}_{i\in\omega}$ be a list of all cl-tt maps. Combining the above strategy for each T_i, we can construct a sequence X such that $\lim_{n\to\infty} M(X \upharpoonright n) = \infty$ and for each i, $T_i^{-1}(X)$ is countable.

□

Thank you for your attention!