Extending CL-Reducibility on Array Noncomputable Degrees

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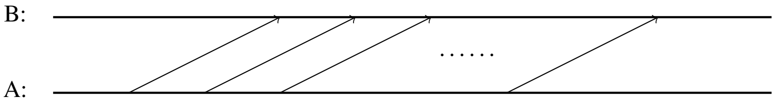
cl-reducibility

Definition 1 (cl-reducibility)

A set A is *cl (computably Lipschitz)-reducible* to a set B (written $A \leq_{\mathrm{cl}} B$), or we say B cl-computes A, if there is a constant c such that A is Turing reducible to B and the use function is bounded from above by $\mathrm{id} + c$.



- If $A \leq_{\mathrm{cl}} B$, then we can describe $A \upharpoonright n$ using $B \upharpoonright n$ and $\mathbf{O}(1)$ many more bits.
- It implies $K(A \upharpoonright n) \leq K(B \upharpoonright n) + \mathbf{O}(1)$.
- Thus, cl-reducibility preserves non-randomness.

cl-reducibility

However, there is no cl-complete left-c.e. real.

Theorem 2 (Yu & Ding, 2004)

There are two left-c.e. reals such that no left-c.e. real cl-computes both of them.

Theorem 3 (Barmpalias & Lewis, 2006)

There is a left-c.e. real not cl-computable from any left-c.e. random real.

cl-reducibility on a.n.c. degrees

Which degrees exactly contain the witnesses of these Theorems?

Theorem 4 (Barmpalias & Downey & Greenberg, 2010)

The following are equivalent for a c.e. Turing degree $\mathbf{d}$.

1. *There are two left-c.e. reals in* $\mathbf{d}$ *such that no left-c.e. real cl-computes both of them.*
2. *There is a left-c.e. real in* $\mathbf{d}$ *such that no left-c.e. random real cl-computes it.*
3. $\mathbf{d}$ *is array noncomputable.*

weakening cl-reducibility on a.n.c. degrees

Observation: For Turing reducibility in general, every left-c.e. random real is Turing complete, thus Turing computes all left-c.e. reals.

Definition 5 (bounded Turing reducibility)

A set A is *f-bounded-Turing (f-bT-) reducible* to a set B (written $A \leq_{f-\mathrm{bT}} B$), or we say B f-bT-computes A, if A is Turing reducible to B and the use function is bounded from above by f.

Cl-reducibility can be seen as the uniform version of $(\mathrm{id}+c)$-bT-reducibility for a constant c. (We will come back to this later.)

Question:

If we weaken the reducibility by allowing "larger" use functions, for which functions exactly the previous theorem still holds?

an analogue from the global structure

Kučera-Gács Theorem

Every set is wtt-reducible to a random set.

Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a nondecreasing function.

- It is called *slow-growing* if $\sum_{n\in\omega} 2^{-g(n)} = \infty$;
- it is called *fast-growing* if $\sum_{n\in\omega} 2^{-g(n)} < \infty$.

Theorem 6 (Barmpalias & Lewis-Pye & Teutsch, 2016)

Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a computable slow-growing function. There exists a set that is not $(\mathrm{id}+g)$-bT-reducible to any random set.

Theorem 7 (Barmpalias & Lewis-Pye, 2018)

Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a computable fast-growing function. Every set is $(\mathrm{id}+g)$-bT-reducible to some random set.

the local structure

The local structure is similar to the global one.

Theorem 8 (Barmpalias & F. & Lewis-Pye, 2016)

Every left-c.e. real is $(\mathrm{id}+g)$*-bT-reducible to any left-c.e. random real for any computable fast-growing function* g*.*

Theorem 9 (Barmpalias & F. & Lewis-Pye, 2016; streamlined in F. & Merkle, 2025)

Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ *be a computable slow-growing function. There is a pair of left-c.e. reals such that no left-c.e. real* $(\mathrm{id}+g)$*-bT-computes both of them. Moreover, such a pair exists in every array noncomputable c.e. Turing degree.*

Theorem 10 (F. & Merkle, 2025)

Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ *be a computable slow-growing function. There is a left-c.e. real such that no left-c.e. random real* $(\mathrm{id}+g)$*-bT-computes it. Moreover, such a real exists in every array noncomputable c.e. Turing degree.*

The degree characterizations still hold.

Corollary 11

Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a computable slow-growing function. The following are equivalent for a c.e. Turing degree $\mathbf{d}$.

1. *There are two left-c.e. reals in $\mathbf{d}$ such that no left-c.e. real $(\mathrm{id}+g)$-bT-computes both of them.*
2. *There is a left-c.e. real in $\mathbf{d}$ such that no left-c.e. random real $(\mathrm{id}+g)$-bT-computes it.*
3. *$\mathbf{d}$ is array noncomputable.*

basic proof idea of Theorem 9

Lemma 12

Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a computable slow-growing function. There is a strictly increasing computable function ℓ such that the following hold. For any triple $\langle \Phi, \Psi, \gamma \rangle$, where Φ, Ψ are Turing functionals with use functions bounded by $\mathrm{id} + g$ and γ is a left-c.e. real in $[0,1)$, uniformly in $\langle \Phi, \Psi, \gamma \rangle$ and in all $a \in \mathbb{N}$, there are left-c.e. reals $\alpha_a, \beta_a \subseteq [a, a+\ell(a)]$ such that

$$\exists\, x \in [a, a+\ell(a)] \left(\alpha_a(x) \neq \Phi^{\gamma}(x) \vee \beta_a(x) \neq \Psi^{\gamma}(x) \right). \tag{1}$$

Basic strategy: Let $b = a + \ell(a)$.

- Whenever (1) fails, we add 2^{-b} to α_a. Suppose the least changed position of α_a is n.
- If (1) fails again, γ should have changed at position $\leq n + g(n)$. Then we do the same for β_a.
- If (1) fails once again, γ should have changed at position $\leq n + g(n)$ again. After one round, it changed at position $\leq n + g(n)$ twice, which makes it grow at least $2^{-n-g(n)}$.
- By choosing b appropriately, we can force γ to be larger than 1 while only change bits of α_a, β_a within $[a, b]$.

Using Lemma 12 as modules, we can easily construct one witness as required in Theorem 9. With tools to be introduced later, it also follows directly that such witnesses actually exist in every a.n.c. degree.

basic proof idea of Theorem 10

Lemma 13

Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a computable slow-growing function. There is a strictly increasing computable function ℓ such that the following hold. For any pair $\langle \Phi, \gamma \rangle$, where Φ is a Turing functional with use function bounded by $\mathrm{id} + g$ and γ is a left-c.e. real in $[0, 1)$, uniformly in $\langle \Phi, \gamma \rangle$ and in all $a \in \mathbb{N}$, there are a left-c.e. real $\alpha_a \subseteq [a, a + \ell(a)]$ and a c.e. set E_a of strings with $\mu(E_a) < 2^{-a}$ such that

$$\exists\, x \in [a, a + \ell(a)]\ \alpha_a(x) \neq \Phi^{\gamma}(x) \quad \textit{or} \quad \gamma \in [\![E_a]\!]. \tag{2}$$

- Instead of having two reals α_a, β_a at disposal, now we have a left-c.e. real α_a and a c.e. set E_a.
- With some modification, we can let E_a act in the place where previously β_a should act.

Array noncomputablity - definition and basic properties

- A sequence $\mathcal{F} = \{F_n\}_{n\in\omega}$ of finite sets is a *very strong array (v.s.a.)* if the following hold.
  1. There is a computable function f such that $f(n)$ is the canonical index of F_n;
  2. $\bigcup_{n\in\omega} F_n = \mathbb{N}$;
  3. $F_n \cap F_m = \emptyset$ for $n \neq m$;
  4. $0 < |F_n| < |F_{n+1}|$ for all $n \in \omega$.
- A c.e. set A is $\mathcal{F}$*-a.n.c.* if it is $\mathcal{F}$*-similar* to any c.e. set V, i.e.,

$$\exists^\infty n \left(A \cap F_n = V \cap F_n\right)$$

- A c.e. set A is *a.n.c.* if it is $\mathcal{F}$-a.n.c. for some v.s.a. $\mathcal{F}$.
- A c.e. degree $\mathbf{a}$ is *array noncomputable* if it contains a a.n.c. c.e. set.

Theorem 14 (Downey & Jockusch & Stob, 1990)

Let $\mathbf{d}$ *be an array noncomputable c.e. degree and let* $\mathcal{F}$ *be a very strong array. There is a c.e. set* $D \in \mathbf{d}$ *such that* D *is* $\mathcal{F}$*-a.n.c.*

A.n.c. degrees are those in which the multiple permitting argument can be effectively applied.

Array noncomputability for locally l.c.e. sets

Let $\mathcal{F} = \{F_n\}_{n\geq 0}$ be a very strong array of intervals.

- A left-c.e. approximation $\{\alpha_s\}_{s\geq 0}$ is $\mathcal{F}$*-compatible* if $\alpha_s \upharpoonright F_n \leq_{lex} \alpha_{s+1} \upharpoonright F_n$ for all $n, s \geq 0$ and $\alpha_s(x) \leq \alpha_{s+1}(x)$ for all $s \geq 0$ and $x \notin \bigcup_{n\geq 0} F_n$.
- A real α is $\mathcal{F}$*-compatibly left-c.e.* ($\mathcal{F}$*-l.c.e.* for short) if it has a $\mathcal{F}$-compatible left-c.e. approximation.
- An $\mathcal{F}$-l.c.e. real α is $\mathcal{F}$*-l.c.e.-a.n.c.* if it is $\mathcal{F}$-similar to any $\mathcal{F}$-l.c.e. real β, i.e.,

$$\exists^\infty n \; (\alpha \upharpoonright F_n = \beta \upharpoonright F_n).$$

Theorem 15

Let $\mathbf{d}$ *be an array noncomputable c.e. degree and let* $\mathcal{F}$ *be a very strong array of intervals. There is a left-c.e. real* $\alpha \in \mathbf{d}$ *such that* α *is* $\mathcal{F}$*-l.c.e.-a.n.c.*

l.c.e. ℓ-property

Definition 16

Let ℓ be a strictly increasing computable function. A property $\mathcal{P}$ is an *l.c.e. ℓ-property* if there are uniformly left-c.e. reals $\{\beta_a\}_{a\in\omega}$ with $\beta_a \subseteq [a, a+\ell(a)]$ such that for any real γ

$$\exists^\infty a \; (\gamma \upharpoonright [a, a+\ell(a)] = \beta_a) \;\Rightarrow\; \gamma \in \mathcal{P}. \tag{3}$$

Theorem 17

Let ℓ be a strictly increasing computable function and let **d** *be an array non-computable c.e. degree. There is a left-c.e. real in* **d** *which has all l.c.e. ℓ-properties $\mathcal{P}$.*

proof of Theorem 10

Lemma 18

Let $g \colon \mathbb{N} \mapsto \mathbb{N}$ be a computable nondecreasing slow-growing function. There is a strictly increasing computable function ℓ such that, for any pair $\langle \Phi, \gamma \rangle$, where Φ is a Turing functional with use function bounded by $\mathrm{id}+g$ and γ is a left-c.e. real in $[0,1]$, the property

$$\mathcal{P}_{\Phi,\gamma} = \{\alpha \colon \alpha \neq \Phi^\gamma \text{ or } \gamma \text{ is not random}\}$$

is an almost-c.e. ℓ-property.

Proof.

- Let ℓ be the function as given by Lemma 13. Then for any pair $\langle \Phi, \gamma \rangle$, there are $\{\alpha_a\}_{a \geq 0}$ and $\{E_a\}_{a \geq 0}$ as stated in Lemma 13.
- Given any left-c.e. real $\tilde{\alpha}$, suppose $\exists^\infty a$ such that $(\tilde{\alpha} \upharpoonright [a, a+\ell(a)] = \alpha_a)$.
- Suppose $\tilde{\alpha} = \Phi^\gamma$. Then it suffices to show that γ is not random.
- By (2), it follows that $\exists^\infty a$ such that $\gamma \in [E_a]$.
- Let $U_e = \bigcup_{a>e} [E_a]$. Then $\gamma \in U_e$ for all $e \geq 0$.
- On the other hand, U_e are uniformly c.e., and $\mu(U_e) \leq \sum_{a>e} \mu(E_a) < \sum_{a>e} 2^{-a} \leq 2^{-e}$.
- Thus, $\{U_e\}_{e \in \omega}$ is a Martin-Löf test containing γ.

$\square$

proof of Theorem 10

Theorem 10

Let $g\colon \mathbb{N} \mapsto \mathbb{N}$ be a computable slow-growing function. There is a left-c.e. real such that no left-c.e. random real $(\mathrm{id}+g)$-bT-computes it. Moreover, such a real exists in every array noncomputable c.e. Turing degree.

Proof.

- Let $\{\Phi_i, \gamma_i\}_{i\in\omega}$ be an effective enumeration of all pairs of a Turing functional with use function bounded by $\mathrm{id}+g$ and a left-c.e. real in the unit interval.
- By Lemme 18, there is a strictly increasing computable function ℓ such that, for all i the properties $\mathcal{P}_i = \{\alpha\colon \alpha \neq \Phi_i^{\gamma_i} \text{ or } \gamma_i \text{ is not random}\}$ are almost-c.e. ℓ-properties.
- By Theorem 17, there is a left-c.e. real $\alpha \in \mathbf{d}$ such that $\alpha \in \mathcal{P}_i$ for all i, which implies it is not $(\mathrm{id}+g)$-bounded Turing reducible to any left-c.e. random real.

□

a.n.c. degrees and not totally ω-c.e. degrees

- An *order* is a non-decreasing and unbounded function.
- Given a function $f \leq_T 0'$, let $\{f_s\}$ be a computable approximation of f. The *mind-change function* of $\{f_s\}$ is defined as $m_f(n) = \#\{s : f_s(n) \neq f_{s+1}(n)\}$.
- Given a computable order h, a function f is *h-c.e.* if it has some computable approximation such that h bounds the mind-change function of that approximation.

Another characterization of a.n.c. degrees

A c.e. degree $\mathbf{d}$ is a.n.c. if for every computable order h there is a function $f \in \mathbf{d}$ which is not h-c.e.

By switching the quantifiers, we get the uniform analogue of a.n.c. degrees.

Definition 19

A c.e. degree $\mathbf{d}$ is *not totally ω-c.e.* if there is a function $f \in \mathbf{d}$ which is not h-c.e. for every computable order h.

The following theorem is a uniform analogue fo Theorem 17.

Theorem 20 (Ambos-Spies & Losert & Monath)

Let $\mathbf{d}$ be a c.e. Turing degree which is not totally ω-c.e.. There is a left-c.e. real in $\mathbf{d}$ which has all l.c.e. ℓ-properties for any strictly increasing computable function ℓ.

wcl-reducibility on not totally ω-c.e. degrees

We can define a uniform version of $(\mathrm{id}+g)$-bT-reducibility for slow-growing function g.

Definition 21 (wcl-reducibility)

A set A is *wcl (weakly computably Lipschitz)-reducible* to a set B (written $A \leq_{\mathrm{wcl}} B$), or we say B wcl-computes A, if there is a slow-growing function g such that A is Turing reducible to B and the use function is bounded from above by $\mathrm{id} + g$.

From Theorem 20, Lemma 12 and Lemma 13, we get the following corollary as a uniform analogue of Corollary 11.

Corollary 22

The following are equivalent for a c.e. Turing degree **d**.

1. *There are two left-c.e. reals in* **d** *such that no left-c.e. real wcl-computes both of them.*
2. *There is a left-c.e. real in* **d** *such that no left-c.e. random real wcl-computes it.*
3. **d** *is not totally ω-c.e.*

Thank you for your attention!