Chaitin's constant and its properties

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What is Chaitin's constant

Chaitin's constant Ω is defined as follows.

$$\Omega = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|},$$

where U is a universal prefix-free Turing machine.

- U is prefix-free if its domain is prefix-free, i.e. ,

$$\forall \sigma, \tau \in \mathrm{dom}(U)[\sigma \not\prec \tau].$$

- U is universal if it can simulate any other prefix-free Turing machine, i.e. , given an effective list of all prefix-free machines $\{M_e\}$ there exists a computable function $e \mapsto \sigma_e$ such that

$$\forall \sigma \in 2^{<\omega}[U(\sigma_e^\frown \sigma) = M_e(\sigma)].$$

Universal prefix-free Turing machine exists

Let $\{M_e\}$ be an effective list of all prefix-free machines $\{M_e\}$. Define a machine V as follows.

$$V(0^e 1^\frown \sigma) = M_e(\sigma) \text{ for all } \sigma \in 2^{<\omega}.$$

Then V is a universal prefix-free Turing machine.

Ω as halting probability

$$\Omega = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|}$$

The probability measure on Cantor space is defined so that for any binary string σ the set of sequences that begin with σ has measure $2^{-|\sigma|}$.

Taking this probability measure on Cantor space, Ω represents the probability that a randomly selected infinite sequence contains a prefix that the machine U will halt on.

$\Omega =_T 0'$

Fact:

Knowing the first N bits of Ω, one could compute the halting problem for all programs of a size up to N.

- Let
$$\Omega_s = \sum_{|\sigma|<s \& U[s](\sigma)\downarrow} 2^{-|\sigma|}.$$
- Find s such that $\Omega_s \upharpoonright N = \Omega \upharpoonright N$.
- Then all programs of size less than N that has not halted at step s will never halt.
- Because otherwise, if some program of size less than N halts later, it will contribute at least 2^{-N} to Ω, which makes $\Omega_s \upharpoonright N < \Omega \upharpoonright N$.

Thus, Ω is Turing equivalent to the halting problem and therefore it is not computable.

Ω is left-c.e.

Definition

Let $\{\alpha_s\}_{s\in\omega}$ be a computable nondecreasing sequence of rationals converging to α. We say that α is a left-c.e. real and $\{\alpha_s\}_{s\in\omega}$ is a left-c.e. approximation of α.

We can define right-c.e. reals and approximations similarly.

Though Ω is not computable, it is left-c.e. $\{\Omega_s\}_{s\in\omega}$ is a left-c.e. approximation of it.

$$\Omega_s = \sum_{|\sigma|<s \& U[s](\sigma)\downarrow} 2^{-|\sigma|}$$

Ω is Martin-Löf random

Theorem (Chaitin 1975)

Ω is Martin-Löf random.

Martin-Löf random can be defined in several ways. A sequence is Martin-Löf random if it is

- typical: it is not contained in any effective null class.
- incompressible: the shortest program to output the first n bits of it must be of size at least $n - \mathbf{O}(1)$.
- unpredictable: no effective way to predict the next bit given the previous bits.

As a consequence, Ω is a normal number.

A characterization for Ω

Definition

A left-c.e. real α is *Solovay reducible* to a left-c.e. real β ($\alpha \leq_S \beta$) if there are a left-c.e. approximation $\{\alpha_s\}_{s\in\omega}$ of α, a left-c.e. approximation $\{\beta_s\}_{s\in\omega}$ of β and a constant c such that $\alpha - \alpha_s \leq c(\beta - \beta_s)$ for all $s \in \omega$.

A left-c.e. real is *Solovay complete* if every left-c.e. real is Solovay reducible to it.

With the works by Chaitin (1975), Solovay (1975), Calude, Hertling, Khoussainov, and Wang (2001), and Kucera and Slaman (2001), it has been shown that the following are equivalent.

1. α is a random left-c.e. real;
2. α is the halting probability of a universal prefix-free machine;
3. α is Solovay complete.

d.c.e. reals

Definition

A real is *d.c.e.* if it is the difference of two left-c.e. reals.

Proposition (Ambos-Spies, Weihrauch, Zheng 2000)

A real α is d.c.e. if and only if there is a computable sequence of rationals $\{\alpha_s\}_{s\in\omega}$ converging to α such that the variation $\sum_{s\in\omega}|\alpha_{s+1}-\alpha_s|$ is bounded.

Definition

A real is Δ^0_2 if there is a computable sequence of rationals converging to it.

Random d.c.e. reals have the following nice property.

Proposition (Rettinger, Zheng 2005)

Random d.c.e. reals are either left-c.e. reals or right-c.e. reals.

convergence rate of d.c.e. reals

Fix a left-c.e. approximation $\{\Omega_s\}_{s\in\omega}$ of Ω.

Definition

For a d.c.e. real α with d.c.e. approximations $\{\alpha_s\}_{s\in\omega}$, let

$$\partial\alpha = \partial\{\alpha_s\} = \lim_{s\to\infty} \frac{\alpha - \alpha_s}{\Omega - \Omega_s}.$$

Theorem (Barmpalias, Lewis-Pye 2017; refined by Miller 2017)

Let α be a d.c.e. real.

1. $\partial\alpha$ *converges and is independent of the choice of* $\{\alpha_s\}_{s\in\omega}$.
2. $\partial\alpha = 0$ *if and only if* α *is not random.*
3. $\partial\alpha > 0$ *if and only if* α *is random and left-c.e.*
4. $\partial\alpha < 0$ *if and only if* α *is random and right-c.e.*

Consequently, left-c.e. approximations to any Ω converges the slowest amongst all left-c.e. approximations.

list of notions of speedability

Definition 1

Let $\rho \in [0, 1)$ be some rational and α some real.

- Suppose α has a left-c.e./d.c.e./Δ^0_2 approximation $\{a_s\}_{s\in\omega}$. $\{a_s\}_{s\in\omega}$ is *ρ-speedable* if there is a nondecreasing computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\liminf_{s\to\infty} \left| \frac{\alpha - a_{f(s)}}{\alpha - a_s} \right| \leq \rho. \tag{1}$$

- α is *left-c.e./d.c.e./Δ^0_2 ρ-speedable* if it has one ρ-speedable left-c.e./d.c.e./Δ^0_2 approximation.
- α is *fully left-c.e./d.c.e./Δ^0_2 ρ-speedable* if it is left-c.e./d.c.e./Δ^0_2 and all of its left-c.e./d.c.e./Δ^0_2 approximations are ρ-speedable.
- α is *weakly left-c.e./d.c.e./Δ^0_2 ρ-speedable* if it has two left-c.e./d.c.e./Δ^0_2 approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ such that

$$\liminf_{s\to\infty} \left| \frac{\alpha - b_s}{\alpha - a_s} \right| \leq \rho. \tag{2}$$

- α is *(fully/weakly) left-c.e./d.c.e./Δ^0_2 speedable* if it is (fully/weakly) left-c.e./d.c.e./Δ^0_2 ρ-speedable for some $\rho \in [0, 1)$.

a characterization of random d.c.e. reals

Theorem 2 (Barmpalias, F., Merkle, Titov)

For a d.c.e. real γ, the following are equivalent.

1. *γ is d.c.e. 0-speedable.*
2. *γ is d.c.e. speedable.*
3. *γ is weakly d.c.e. 0-speedable.*
4. *γ is weakly d.c.e. speedable.*
5. *γ is nonrandom.*

Lemma 3

Given a random d.c.e. real γ, for any of its two d.c.e. approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$,

$$\lim_{s\to\infty} \frac{\gamma - b_s}{\gamma - a_s} = 1.$$

Proof.

As γ be a d.c.e. random real, then $\partial\{a_s\} = \partial\{b_s\} = \partial\gamma \neq 0$, i.e. ,

$$\lim_{s\to\infty} \frac{\gamma - a_s}{\Omega - \Omega_s} = \lim_{s\to\infty} \frac{\gamma - b_s}{\Omega - \Omega_s} \neq 0.$$

Then

$$\lim_{s\to\infty} \frac{\gamma - b_s}{\gamma - a_s} = \lim_{s\to\infty} \frac{\frac{\gamma - b_s}{\Omega - \Omega_s}}{\frac{\gamma - a_s}{\Omega - \Omega_s}} = 1.$$

□

a characterization of random d.c.e. reals

Lemma 4

Given a Δ^0_2 real γ, it has a computable approximation $\{a_s\}_{s\in\omega}$ such that

$$\liminf_{s\to\infty}\left|\frac{\gamma - a_{s+1}}{\gamma - a_s}\right| = 0. \tag{3}$$

Moreover, if γ is d.c.e. , $\{a_s\}_{s\in\omega}$ can also be made d.c.e. .

speedability of Δ^0_2 reals

Let LC, RC, WC, CA denote the set of all left-c.e. , right-c.e. , d.c.e. , Δ^0_2 reals respectively.

Theorem 5 (Barmpalias, F., Merkle, Titov)

Every real in $\mathsf{CA}/(\mathsf{LC} \cup \mathsf{RC})$ is fully Δ^0_2 ρ-speedable for every $\rho \in (0, 1)$.

Corollary 6

Every real in $\mathsf{WC}/(\mathsf{LC} \cup \mathsf{RC})$ is fully d.c.e. ρ-speedable for every $\rho \in (0, 1)$.

Theorem 7 (Barmpalias, F., Merkle, Titov)

Every Δ^0_2 real is Δ^0_2 speedable.

All follow from the following lemma.

Lemma 8

Every two-sided computable approximation is ρ-speedable for every $\rho \in (0, 1)$.

An approximation $\{a_s\}_{s\in\omega}$ to some real α is called *two-sided* if there are infinitely many s such that $a_s < \alpha$ and infinitely many s such that $a_s > \alpha$.

characterize Ω by speedability?

Clearly, we also have:

For a left-c.e. real, if it is random, then it is not (weakly) left-c.e. speedable.

We hoped that the converse is also true. But Hölzl and Janicki showed that there is a non-random left-c.e. real which is not speedable.

Question:

Which randomness notion characterizes the unspeedability of Ω?

speedup functions for approximating Ω

Back to Ω, we failed to characterize it by the speedability of its approximations. However, we can still ask the following question.

Question

Which functions speedup Ω exactly?

Theorem 9

For a strictly increasing function $f : \mathbb{N} \mapsto \mathbb{N}$, if f is dominated by a function g which is computable from some low for Ω set X, then

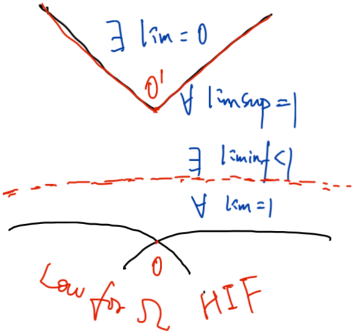
$$\lim_{s \to \infty} \frac{\Omega - \Omega_{f(s)}}{\Omega - \Omega_s} = 1.$$

Conjecture

The following are equivalent for a Turing degree $\mathbf{d}$:

- $\mathbf{d}$ does not compute any speedup function for Ω;
- $\mathbf{d}$ is HIF relative to some low for Ω degree.

speedup functions for approximating Ω



- As $\Omega \equiv_T 0'$, any Turing degree $\geq_T 0'$ contains a function f such that $\lim_{s\to\infty} \frac{\Omega-\Omega_{f(s)}}{\Omega-\Omega_s} = 0$.
- We can also prove that, for any strictly increasing function $f \not\geq_T 0'$, $\limsup_{s\to\infty} \frac{\Omega-\Omega_{f(s)}}{\Omega-\Omega_s} = 1$.
- Our corollary above tells that if a strictly increasing function f is dominated by a low for Ω function, then $\lim_{s\to\infty} \frac{\Omega-\Omega_{f(s)}}{\Omega-\Omega_s} = 1$.
- However, at the moment we still don't know which degrees exactly contain an increasing function f such that $\liminf_{s\to\infty} \frac{\Omega-\Omega_{f(s)}}{\Omega-\Omega_s} < 1$.

Thank you for your attention!