Speedability of computably approximable reals and their approximations

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Chaitin's constant Ω

Chaitin's constant Ω is defined as follows.

$$\Omega = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|},$$

where U is a universal prefix-free Turing machine.

- U is prefix-free if its domain is prefix-free, i.e. ,

$$\forall \sigma, \tau \in \mathrm{dom}(U)[\sigma \not\prec \tau].$$

- U is universal if it can simulate any other prefix-free Turing machine, i.e. , given an effective list of all prefix-free machines $\{M_e\}$ there exists a computable function $e \mapsto \sigma_e$ such that

$$\forall \sigma \in 2^{<\omega}[U(\sigma_e^\frown \sigma) = M_e(\sigma)].$$

The existence of such a universal prefix-free Turing machine is witnessed by

$$V(0^e 1^\frown \sigma) = M_e(\sigma) \text{ for all } \sigma \in 2^{<\omega},$$

where $\{M_e\}$ is an effective list of all prefix-free machines $\{M_e\}$.

Properties of Ω

- Ω is the halting probability of the universal prefix-free Turing machine U
- Ω is Martin-Löf random
- $\Omega =_T 0'$
- Ω is left-c.e. witnessed by

$$\Omega_s = \sum_{|\sigma|<s \& U[s](\sigma)\downarrow} 2^{-|\sigma|}$$

Definition

A real α is *left-c.e.* if there is a computable nondecreasing sequence $\{\alpha_s\}_{s\in\omega}$ of rationals converging to α. $\{\alpha_s\}_{s\in\omega}$ is called a left-c.e. approximation of α.

We can define right-c.e. reals and approximations similarly.

Ω and Solovay reducibility

Definition

A left-c.e. real α is *Solovay reducible* to a left-c.e. real β ($\alpha \leq_S \beta$) if there are a left-c.e. approximation $\{\alpha_s\}_{s\in\omega}$ of α, a left-c.e. approximation $\{\beta_s\}_{s\in\omega}$ of β and a constant c such that $\alpha - \alpha_s \leq c(\beta - \beta_s)$ for all $s \in \omega$.

In other words, $\alpha \leq_S \beta$ if $\{\alpha_s\}_{s\in\omega}$ converges to α faster than $\{\beta_s\}_{s\in\omega}$ converges to β.

A left-c.e. real is *Solovay complete* if every left-c.e. real is Solovay reducible to it. (It has the slowest converging left-c.e. approximations.)

With the works by Chaitin (1975), Solovay (1975), Calude, Hertling, Khoussainov, and Wang (2001), and Kučera and Slaman (2001), it has been shown that the following are equivalent for left-c.e. reals α:

1. α is a random;
2. α is the halting probability of some universal prefix-free machine;
3. α is Solovay complete for left-c.e. reals.

speedability of computable approximations

Definition 1

Let $\{a_s\}_{s\in\omega}$ be some computable approximation converges to α and ρ be some rational in $[0, 1)$. We say $\{a_s\}_{s\in\omega}$ is ρ-*speedable* if there is a nondecreasing computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\liminf_{s\to\infty} \left| \frac{\alpha - a_{f(s)}}{\alpha - a_s} \right| \leq \rho. \tag{1}$$

And we say $\{a_s\}_{s\in\omega}$ is *speedable* if it is ρ-speedable for some $\rho \in (0, 1)$.

Lemma (Merkle, Titov 2020)

If $\{a_s\}_{s\in\omega}$ *is* ρ*-speedable for some* $\rho \in (0, 1)$*, then it is* ρ*-speedable for all* $\rho \in (0, 1)$.

Definition 2

Let α be a c.a. real.

- α is *left-c.e. speedable* if it has a speedable left-c.e. approximation.
- α is *fully left-c.e. speedable* if it is left-c.e. and all of its left-c.e. approximations are speedable.
- α is *weakly left-c.e. speedable* if there exists a rational $\rho \in (0, 1)$ and two left-c.e. approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of α such that

$$\liminf_{s\to\infty} \left| \frac{\alpha - b_s}{\alpha - a_s} \right| \le \rho. \tag{2}$$

- α is *left-c.e. 0-speedable* if it has a speedable left-c.e. approximation which is 0-speedable.

The corresponding notions for right-c.e. , d.c.e. and c.a. are defined likewise.

weak speedability implies speedability

Theorem 3

A real is left-c.e./d.c.e./c.a. speedable if and only if it is weakly left-c.e./d.c.e./c.a. speedable.

Proof.

"$\Rightarrow$:" This direction is obvious. Indeed, if some approximation $\{a_s\}_{s\in\omega}$ is speedable via some function f, $\{a_s\}_{s\in\omega}$ and $\{a_{f(s)}\}_{s\in\omega}$ would then achieve weak speedability.

"$\Leftarrow$:"

- Suppose α is weakly left-c.e. ρ-speedable via two left-c.e. approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$. We define a function $f\colon \mathbb{N} \to \mathbb{N}$ by letting $f(n)$ be the least number such that $a_{f(n)} \geq b_n$. It is easy to check that f is nondecreasing and α is ρ-speedable via f.
- Suppose α is weakly c.a. ρ-speedable via two computable approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$. Define $\{c_s\}_{s\in\omega}$ to be the approximation $\{a_0, b_0, a_1, b_1, \ldots\}$. Clearly, this is also a computable approximation of α, and it is ρ-speedable via the function $f(n) = n + 1$.
- It is difficult to directly get d.c.e.speedability from weak d.c.e.speedability, whereas we provide an indirect proof later.

□

speedability of left-c.e. reals

Theorem (Merkle, Titov 2020)

For a left-c.e. real γ, the following are equivalent.

1. *γ is weakly left-c.e. speedable.*
2. *γ is left-c.e. speedable.*
3. *γ is fully left-c.e. speedable.*

Random left-c.e. reals are never speedable.

Questions:

Q1: is it true that all nonrandom left-c.e. reals are left-c.e. speedable?

Q2: is it true that all left-c.e. speedable reals are left-c.e. 0-speedable?

- Q1 is answered in the negative by Hölzel and Janicki 2023.
- Q2 is answered in the negative by Hölzel, Janicki, Merkle and Stephan 2024.

d.c.e. reals

Definition

A real is *d.c.e.* if it is the difference of two left-c.e. reals.

Proposition (Ambos-Spies, Weihrauch, Zheng 2000)

A real α is d.c.e. if and only if there is a computable sequence of rationals $\{\alpha_s\}_{s\in\omega}$ converging to α such that the variation $\sum_{s\in\omega}|\alpha_{s+1} - \alpha_s|$ is bounded.

Random d.c.e. reals have the following nice property.

Proposition (Rettinger, Zheng 2005)

Random d.c.e. reals are either left-c.e. reals or right-c.e. reals.

derivation on the d.c.e. reals

Fix a left-c.e. approximation $\{\Omega_s\}_{s\in\omega}$ of Ω.

Definition

For a d.c.e. real α with d.c.e. approximations $\{\alpha_s\}_{s\in\omega}$, let

$$\partial\alpha = \partial\{\alpha_s\} = \lim_{s\to\infty} \frac{\alpha - \alpha_s}{\Omega - \Omega_s}.$$

Theorem (Barmpalias, Lewis-Pye 2017; refined by Miller 2017)

Let α be a d.c.e. real.

1. $\partial\alpha$ *converges and is independent of the choice of* $\{\alpha_s\}_{s\in\omega}$.
2. $\partial\alpha = 0$ *if and only if* α *is not random.*
3. $\partial\alpha > 0$ *if and only if* α *is random and left-c.e.*
4. $\partial\alpha < 0$ *if and only if* α *is random and right-c.e.*

random d.c.e. reals are not weakly d.c.e. speedable

Lemma 4

Random d.c.e. reals are not weakly d.c.e. speedable.

Proof.

Let γ be a random d.c.e. real, $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ two d.c.e. approximations of γ. Then $\partial\{a_s\} = \partial\{b_s\} = \partial\gamma \neq 0$, i.e. ,

$$\lim_{s\to\infty} \frac{\gamma - a_s}{\Omega - \Omega_s} = \lim_{s\to\infty} \frac{\gamma - b_s}{\Omega - \Omega_s} \neq 0.$$

Then

$$\lim_{s\to\infty} \frac{\gamma - b_s}{\gamma - a_s} = \lim_{s\to\infty} \frac{\frac{\gamma-b_s}{\Omega-\Omega_s}}{\frac{\gamma-a_s}{\Omega-\Omega_s}} = 1.$$

□

nonrandom d.c.e. reals are d.c.e. 0-speedable

Lemma 5

Given a nonrandom c.a. real γ, it has a computable approximation $\{a_s\}_{s\in\omega}$ such that

$$\liminf_{s\to\infty}\left|\frac{\gamma - a_{s+1}}{\gamma - a_s}\right| = 0. \tag{3}$$

Moreover, if γ is d.c.e. , $\{a_s\}_{s\in\omega}$ can also be made d.c.e. .

Proof.

As γ is not random, let $\{U_i\}_{i\in\omega}$ be a Martin-Löf test which covers γ, i.e. ,

$$\mu(U_i) \leq 2^{-i} \text{ and } \gamma \in [\![U_i]\!] \text{ for all } i \in \omega.$$

We construct an approximation $\{a_s\}_{s\in\omega}$ of γ such that

$$\forall n\, \exists s \geq n \left[\left|\frac{\gamma - a_{s+1}}{\gamma - a_s}\right| \leq 2^{-n}\right], \tag{4}$$

which then implies (3).

Proof (Cont.)

Construction:

- Let $\sigma_i \uparrow$ for all $i \in \omega$.
- At stage $s \geq 0$, find the least integer $n \leq s$ such that $\sigma_n \uparrow$.
- If there exists an index $j < n$ such that $\sigma_j \downarrow \not\prec r_s$, let j be the least such index and $\sigma_i \uparrow$ for all $i \geq j$. Then let $b_s = r_s$ and go to stage $s+1$.
- Otherwise, search for the least $t \geq s$ such that there exists $\sigma \in U_{2n}$ with $\sigma \prec r_t$. If σ has not been marked as *used* in U_{2n}, we mark it as used in U_{2n} at stage t and let $b_t = r_t - 2^{n+1-|\sigma|}$; otherwise, let $b_t = r_t$. Then let $\sigma_n \downarrow = \sigma$ and $b_i = r_i$ for all $s \leq i < t$ and go to stage $t+1$.
- Let $a_{2s} = b_s, a_{2s+1} = r_s$ for all $s \geq 0$.

Verification:

1. For each $n \in \omega$, σ_n is defined and is the unique prefix of γ in U_{2n}. And it is used in U_{2n} only once.
2. If σ_n is used in U_{2n} at stage s, then $s \geq n$ and $\left|\frac{\gamma - a_{2s+1}}{\gamma - a_{2s}}\right| \leq 2^{-n}$.
3. $\{a_s\}_{s\in\omega}$ is a computable approximation of γ. If $\{r_s\}_{s\in\omega}$ is d.c.e. , $\{a_s\}_{s\in\omega}$ is d.c.e. as well.

$\square$

speedability of d.c.e. reals

Theorem 6

For a d.c.e. real γ, the following are equivalent.

1. *γ is d.c.e. 0-speedable.*
2. *γ is d.c.e. speedable.*
3. *γ is weakly d.c.e. speedable.*
4. *γ is nonrandom.*

How about full d.c.e. speedability?

The negative answer of Q1 by Hölzel and Janicki 2023 indicates that d.c.e. speedability does not imply full d.c.e. speedability.

However, we can save this for pure d.c.e. reals.

speedability of two-sided computable approximations

An approximation $\{a_s\}_{s\in\omega}$ to some real α is called *two-sided* if there are infinitely many s such that $a_s < \alpha$ and infinitely many s such that $a_s > \alpha$.

Lemma 7

Every two-sided computable approximation is speedable.

A d.c.e. real is *pure* if it is neither left-c.e. nor right-c.e. .

Corollary 8

Every pure d.c.e. real is fully d.c.e. speedable.

Proof idea of Lemma 7 (I)

We prove by contradiction.
Suppose $\{a_s\}_{s\in\omega}$ is a two-sided computable approximable to real α, and $\{a_s\}_{s\in\omega}$ is not ρ-speedable.
Let $c = \rho/(1+\rho)$ and k be an integer such that $k \geq 1 + c^{-2}$. For every $i \in \{0, \cdots, 2k\}$, we inductively define a function $f_i \colon \mathbb{N} \to \mathbb{N}$ and a constant n_i such that

$$\forall n \geq n_i \, \alpha \notin I_i(n), \tag{5}$$

where

$$I_i(n) = \left[a_{f_i(n)} - c|a_{f_i(n)} - a_n|, \, a_{f_i(n)} + c|a_{f_i(n)} - a_n|\right].$$

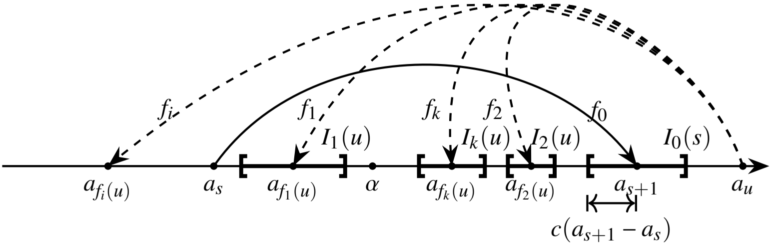
Let $f_0(n) = n + 1$, and n_0 be the index such that $\alpha \notin I_0(n)$.
For $1 \leq i \leq 2k$, inductively, let

$$f_i(n) = \begin{cases} n+1 & \text{if } n < n_{i-1}, \\ \min\{m > n : a_m \notin \bigcup_{j=0}^{i-1} \bigcup_{\ell=n_j}^{n} I_j(\ell)\} & \text{otherwise.} \end{cases} \tag{6}$$

and n_i be the index such that $\alpha \notin I_i(n)$.

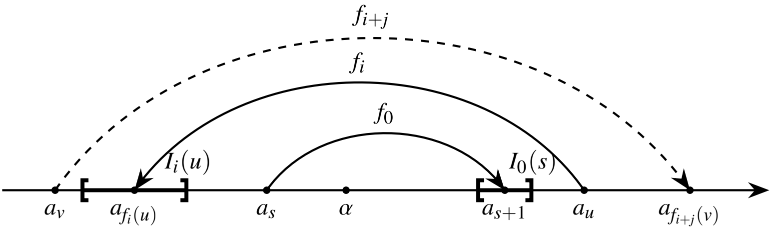
Proof idea of Lemma 7 (II)

- As $\{a_s\}_{s\in\omega}$ is a two-sided approximation to α, let $s \geq n_{2k}$ be an index such that $a_s < \alpha < a_{s+1}$.
- We first find the index $u \geq s+1$ such that a_u is the last one to the right of or equal to a_{s+1}.
- Then all of $a_{f_1(u)}, a_{f_2(u)}, \ldots, a_{f_k(u)}$ are to the left of a_{s+1}.
- Moreover, they must jump over the interval $[a_{s+1} - c(a_{s+1} - a_s), a_{s+1}]$. Then their distances with a_u is bounded below.
- Thus, by our arrangement of the parameters, we argue that at least one of them must be to the left of a_s.



Proof idea of Lemma 7 (III)

- Fixing one such point $a_{f_i(u)}$, we find the index $v \geq f_i(u)$ such that a_v is the last one to the left of or equal to $a_{f_i(u)}$.
- In a symmetrical way as before, we find one point $a_{f_{i+j}(v)}$ to be right of a_u.
- However this contradicts to the fact that a_u is the last one to the right of or equal to a_{s+1}.



Theorem 9

Every c.a. real is c.a. speedable.

- Let $\alpha \in (0,1)$ be a c.a. real with a computable approximation $\{a_s\}_{s\in\omega}$ of α.
- By Lemma 7 it suffices to show that α has a two-sided computable approximation.
- Without loss of generality, assume that a_s differs from a_{s+1} for all s.
- For each $s > 0$, let $h(s)$ be the least position of changed bits of a_s compared to a_{s-1}.
- Let
$$a_s^- = \max\{a_s - 2^{-h(s)}, 0\} \qquad \text{and} \qquad a_s^+ = \min\{a_s + 2^{-h(s)}, 1\},$$
- As $\{a_s\}_{s\in\omega}$ converges to the real α, the values $h(s)$ tend to infinity.
- Furthermore, there are infinitely many stages s where all bits of a_s up to position $h(s)$ are already stable, and for each such stage s, clearly, $a_s^- < \alpha < a_s^+$.
- Then the sequence $\{a_1^-, a_1^+, a_2^-, a_2^+, \dots\}$ is a two-sided computable approximation of α.

speedup functions for approximating Ω

Back to Ω, we failed to characterize it by the speedability of its left-c.e. approximations. However, we can still ask the following question.

Question

Which functions speedup Ω, i.e.,

$$\liminf_{s\to\infty} \frac{\Omega - \Omega_{f(s)}}{\Omega - \Omega_s} < 1?$$

Theorem 10

For a strictly increasing function $f : \mathbb{N} \mapsto \mathbb{N}$, if f is dominated by a function g which is computable from some low for Ω set X, then it does not speedup Ω.

Conjecture

The following are equivalent for a Turing degree $\mathbf{d}$:

- $\mathbf{d}$ does not compute any speedup function for Ω;
- $\mathbf{d}$ is HIF relative to some low for Ω degree.

Thank you for your attention!