

Speedability of computably approximable reals and their approximations

Nan Fang

Institute of Software, Chinese Academy of Sciences

Joint work with *George Barmpalias*, *Wolfgang Merkle*, and *Ivan Titov*

Lanzhou University
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computable approximations and computably approximable reals

Let $\{a_s\}_{s \in \omega}$ be a sequence of rational numbers that converges to α .

- ▶ $\{a_s\}_{s \in \omega}$ is a *computable approximation* (of α) if it is a computable sequence, i.e., there is a Turing machine which, on input s , outputs a_s .
- ▶ $\{a_s\}_{s \in \omega}$ is *left-c.e.* (resp., *right-c.e.*) if it is computable and nondecreasing (resp., nonincreasing).
- ▶ $\{a_s\}_{s \in \omega}$ is *d.c.e.* if it is computable and $\sum_s |a_{s+1} - a_s|$ is finite.
- ▶ A real α is *computably approximable* if it has a computable approximation.
- ▶ *Left-c.e.* reals, *right-c.e.* reals and *d.c.e.* reals are defined likewise.

We study the speedability of all this different classes of reals according to the speedability of their computable approximations.

example: Chaitin's Ω

Fix an universal prefix-free machine U , its halting probability is defined as follows.

$$\Omega = \sum_{U(\sigma) \downarrow} 2^{-|\sigma|}.$$

Let

$$\Omega_s = \sum_{|\sigma| < s \text{ \& } U[s](\sigma) \downarrow} 2^{-|\sigma|},$$

then $\{\Omega_s\}_{s \in \omega}$ is a left-c.e. approximation of Ω .

This number was first defined by Chaitin, who proved that Ω is Martin-Löf random.

In the study of algorithmic randomness, Martin-Löf random sequences can be defined in several equivalent ways.

A sequence is Martin-Löf random if it is

- ▶ typical: it is not contained in any effective null class.
- ▶ incompressible: the shortest program to output the first n bits of it must be of size at least $n - \mathbf{O}(1)$.
- ▶ unpredictable: no effective way to predict the next bit given the previous bits.

Solovay reducibility

For left-c.e. reals α and β , α is *Solovay reducible* to β , denoted $\alpha \leq_S \beta$, if

- (*) there exist a left-c.e. approximation $\{a_s\}_{s \in \omega}$ of α , a left-c.e. approximation $\{b_s\}_{s \in \omega}$ of β , and a constant c such that $\alpha - a_s \leq c(\beta - b_s)$ for all s .

Note that if α is not rational, the condition (*) is equivalent to each of the following two conditions.

- (*') For any left-c.e. approximation $\{a'_s\}_{s \in \omega}$ of α , there exists a left-c.e. approximation $\{b'_s\}_{s \in \omega}$ of β and a constant c such that $\alpha - a'_s \leq c(\beta - b'_s)$ for all s .
- (*'') For any left-c.e. approximation $\{b''_s\}_{s \in \omega}$ of β , there exists a left-c.e. approximation $\{a''_s\}_{s \in \omega}$ of α and a constant c such that $\alpha - a''_s \leq c(\beta - b''_s)$ for all s .

These equivalences indicate that the notion of Solovay reducibility is robust and that, intuitively speaking,

$\alpha \leq_S \beta$ amounts to α being approximable by left-c.e. approximations at least as fast as β .

convergence rate of random left-c.e. reals

The degree structure induced by Solovay reducibility on the class of left-c.e. reals has been well studied. One important result is the following theorem by Solovay, and Kučera and Slaman.

Theorem 1 (Solovay; Kučera & Slaman)

A left-c.e. real is Solovay complete if and only if it is Martin-Löf random.

Here, a left-c.e. real is *Solovay complete* if every left-c.e. real is Solovay reducible to it.

This theorem implies that among left-c.e. reals exactly the random ones converge as slowly as possible.

This phenomenon was further explored by [Barnpalias and Lewis-Pye, 2017] and [Miller, 2017].

a derivation on the d.c.e. reals

Fix a random left-c.e. real Ω with a left-c.e. approximation $\{\Omega_s\}_{s \in \omega}$.

Definition 2

Given a d.c.e. real α with a d.c.e. approximation $\{\alpha_s\}_{s \in \omega}$, let

$$\partial\alpha = \partial\{\alpha_s\} = \lim_{s \rightarrow \infty} \frac{\alpha - \alpha_s}{\Omega - \Omega_s}.$$

By [Barnmpalias and Lewis-Pye, 2017] and [Miller, 2017], $\partial\alpha$ is well-defined and its value is independent of the choice of $\{\alpha_s\}_{s \in \omega}$.

Theorem 3 (Barnmpalias & Lewis-Pye; Miller)

$\partial\alpha = 0$ if and only if α is not random.

convergence rate of random d.c.e. reals

Corollary 4

All the d.c.e. approximations to a random d.c.e. real converge at exactly the same speed.

Proof.

Let γ be a random d.c.e. real, $\{a_s\}_{s \in \omega}$ and $\{b_s\}_{s \in \omega}$ two d.c.e. approximations of γ . Then $\partial\{a_s\} = \partial\{b_s\} = \partial\gamma \neq 0$, i.e. ,

$$\lim_{s \rightarrow \infty} \frac{\gamma - a_s}{\Omega - \Omega_s} = \lim_{s \rightarrow \infty} \frac{\gamma - b_s}{\Omega - \Omega_s} \neq 0.$$

Then

$$\lim_{s \rightarrow \infty} \frac{\gamma - b_s}{\gamma - a_s} = \lim_{s \rightarrow \infty} \frac{\frac{\gamma - b_s}{\Omega - \Omega_s}}{\frac{\gamma - a_s}{\Omega - \Omega_s}} = 1.$$



definition of speedability

Instead of comparing the speed of convergence of the approximations of two different left-c.e. reals, one can also consider

- ▶ the speed of convergence of different approximations of a single left-c.e. real in general,
- ▶ the possibility of speeding up the convergence rate of a single computable approximation.

Definition 5 (speedability for computable approximations)

Let $\{a_s\}_{s \in \omega}$ be a computable approximation of a real α and $\rho \in [0, 1)$ be some rational.

- ▶ If $a_s = \alpha$ for all but finitely many s , we simply say $\{a_s\}_{s \in \omega}$ is ρ -speedable.
- ▶ Otherwise, we say $\{a_s\}_{s \in \omega}$ is ρ -speedable if there is a nondecreasing computable function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\liminf_{s \rightarrow \infty} \left| \frac{\alpha - a_{f(s)}}{\alpha - a_s} \right| \leq \rho.$$

We say $\{a_s\}_{s \in \omega}$ is *speedable* if it is ρ -speedable for some $\rho \in (0, 1)$.

definition of speedability

Definition 6

Let α be a real number, and ρ be some rational. Let X stand for one of the terms left-c.e., right-c.e., d.c.e., or c.a.

- ▶ α is X (ρ -)speedable if it has a (ρ -)speedable X approximation.
- ▶ α is *strongly* X (ρ -)speedable if it has an X approximation that is (ρ -)speedable via the function $s \mapsto s + 1$.
- ▶ α is *fully* X speedable if it is X and all of its X approximations are speedable.
- ▶ α is *weakly* X speedable if there exist a rational $\rho \in (0, 1)$ and two X approximations $\{a_s\}_{s \in \omega}$ and $\{b_s\}_{s \in \omega}$ of α such that

$$\liminf_{s \rightarrow \infty} \left| \frac{\alpha - b_s}{\alpha - a_s} \right| \leq \rho. \quad (1)$$

properties of speedability

For all $X \in \{\text{left-c.e.}, \text{right-c.e.}, \text{d.c.e.}, \text{c.a.}\}$, the following hold.

strong X speedability $\Rightarrow X$ speedability $\Rightarrow$ weak X speedability

full X speedability $\Rightarrow X$ speedability

X 0-speedability $\Rightarrow X$ speedability

Questions:

- ▶ Can any of the above implications be reversed for some X ?
- ▶ What's the relation between speedability and non-randomness?

left-c.e. speedability

[Merkle & Titov, 2020] studied left-c.e. speedability and proved that for left-c.e. in place of $\mathbf{X}$, most of the implications can be reversed.

Theorem 7 (Merkle & Titov)

For a left-c.e. real γ , the following are equivalent.

- ❶ γ is left-c.e. speedable.
- ❷ γ is fully left-c.e. speedable.
- ❸ γ is strongly left-c.e. speedable.
- ❹ γ is weakly left-c.e. speedable.

left-c.e. speedability

By Theorem 3, all left-c.e. random reals are not left-c.e. speedable.

Questions:

Q1: Are all nonrandom left-c.e. reals left-c.e. speedable?

Q2: Are all left-c.e. speedable reals left-c.e. 0-speedable?

- ▶ Q1 is answered in the negative by [Hölzel & Janicki, 2023].
- ▶ Q2 is answered in the negative by [Hölzel, Janicki, Merkle & Stephan, 2024].

d.c.e. speedability

For d.c.e. in place of X, we show that most of the simple implications can be reversed. Moreover, we will prove that a d.c.e. real is d.c.e. speedable if and only if it is nonrandom.

Theorem 8 (Barnaliyas, Merkle, Titov, F.)

For a d.c.e. real γ , the following are equivalent.

- ① γ is d.c.e. 0-speedable.
- ② γ is strongly d.c.e. speedable.
- ③ γ is d.c.e. speedable.
- ④ γ is weakly d.c.e. speedable.
- ⑤ γ is nonrandom.

Lemma 9

If γ is a nonrandom d.c.e. real, then γ is strongly d.c.e. 0-speedable.

full d.c.e. speedability

- ▶ Due to the negative answer of Q1 by [Hölzel & Janicki, 2023], there is a nonrandom left-c.e. real which is not left-c.e. speedable.
- ▶ For this real, it is d.c.e. speedable but not fully d.c.e. speedable, since it has a left-c.e. approximation which is not speedable and all left-c.e. approximations are also d.c.e. approximations.
- ▶ Thus, d.c.e. speedability does not imply full d.c.e. speedability.

However, the implication holds when restricting attention to d.c.e. reals that are neither left-c.e. nor right-c.e. .

Theorem 10 (Barmpalias, Merkle, Titov, F.)

If a d.c.e. real is neither left-c.e. nor right-c.e., then it is fully d.c.e. speedable.

Lemma 11

Every two-sided computable approximation is speedable.

c.a. speedability

Corollary 12

If a c.a. real is neither left-c.e. nor right-c.e., then it is fully c.a. speedable.

Proposition 13

Every c.a. real has a two-sided computable approximation.

Corollary 14

Every c.a. real is c.a. speedable.

With a more involved argument, we can show the following stronger result.

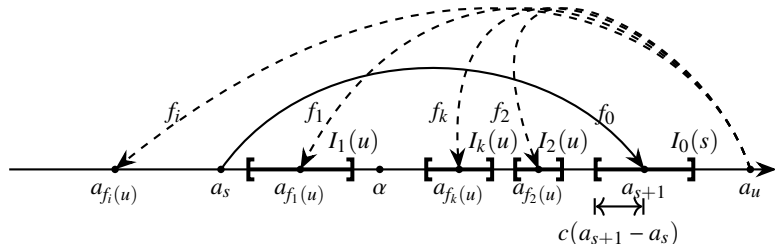
Theorem 15

Every c.a. real is strongly c.a. 0-speedable.

Proof idea of Lemma 11 (I)

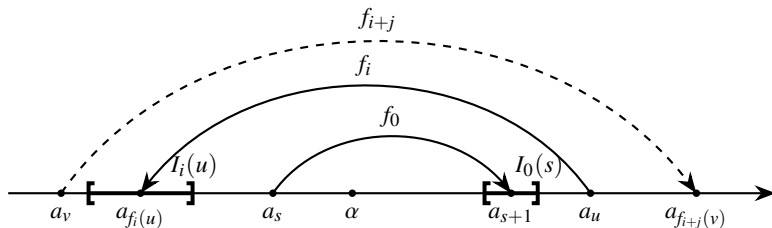
We prove by contradiction. Suppose $\{a_s\}_{s \in \omega}$ is a two-sided computable approximable to real α , and $\{a_s\}_{s \in \omega}$ is not ρ -speedable. Let $s \geq n_{2k}$ be an index such that $a_s < \alpha < a_{s+1}$.

- ▶ We inductively define $2k + 1$ many computable functions $f_i: \mathbb{N} \rightarrow \mathbb{N}$ for $0 \leq i \leq 2k$.
- ▶ We first find the index $u \geq s + 1$ such that a_u is the last one to the right of or equal to a_{s+1} .
- ▶ Then all of $a_{f_1(u)}, a_{f_2(u)}, \dots, a_{f_k(u)}$ are to the left of a_{s+1} .
- ▶ Moreover, they must jump over the interval $[a_{s+1} - c(a_{s+1} - a_s), a_{s+1}]$. Then their distances with a_u is bounded below.
- ▶ Thus, by our arrangement of the parameters, we argue that at least one of them must be to the left of a_s .



Proof idea of Lemma 11 (II)

- ▶ Fixing one such point $a_{f_i(u)}$, we find the index $v \geq f_i(u)$ such that a_v is the last one to the left of or equal to $a_{f_i(u)}$.
- ▶ In a symmetrical way as before, we find one point $a_{f_{i+j}(v)}$ to be right of a_u .
- ▶ However this contradicts to the fact that a_u is the last one to the right of or equal to a_{s+1} .



Thank you for your attention!