Speedability of computably approximable reals and their approximations

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computable approximations and computably approximable reals

Let $\{a_s\}_{s\in\omega}$ be a sequence of rational numbers that converges to α.

- $\{a_s\}_{s\in\omega}$ is a *computable approximation (of α)* if it is a computable sequence, i.e., there is a Turing machine which, on input s, outputs a_s.
- $\{a_s\}_{s\in\omega}$ is *left-c.e.* (resp., *right-c.e.*) if it is computable and nondecreasing (resp., nonincreasing).
- $\{a_s\}_{s\in\omega}$ is *d.c.e.* if it is computable and $\sum_s |a_{s+1} - a_s|$ is finite.
- A real α is *computably approximable* if it has a computable approximation.
- *Left-c.e.* reals, *right-c.e.* reals and *d.c.e.* reals are defined likewise.

We study the speedability of all this different classes of reals according to the speedability of their computable approximations.

example: Chaitin's Ω

Fix an universal prefix-free machine U, the following number is known as its halting probability.

$$\Omega = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|}.$$

Let

$$\Omega_s = \sum_{|\sigma|<s \& U[s](\sigma)\downarrow} 2^{-|\sigma|},$$

then $\{\Omega_s\}_{s\in\omega}$ is a left-c.e. approximation of Ω.

This number was first defined by Chaitin, who proved that Ω is Martin-Löf random.

In the study of algorithmic randomness, Martin-Löf random sequences can be defined in several equivalent ways. One equivalent definition is via *Solovay tests*.

A *Solovay test* is a sequence $\{[\ell_i, r_i]\}_{i\in\omega}$ of nonempty intervals such that

- $\{\ell_i\}_{i\in\omega}$ and $\{r_i\}_{i\in\omega}$ are computable sequences of dyadic rationals in $[0,1]$
- $\sum_i (r_i - \ell_i) < \infty$.

A real γ *fails the Solovay test* if $\gamma \in [\ell_i, r_i]$ for infinitely many i.
γ is random if it passes all Solovay tests.

Solovay reducibility

For left-c.e. reals α and β, α is *Solovay reducible* to β, denoted $\alpha \leq_S \beta$, if

$(*)$ there exist a left-c.e. approximation $\{a_s\}_{s\in\omega}$ of α, a left-c.e. approximation $\{b_s\}_{s\in\omega}$ of β, and a constant c such that $\alpha - a_s \leq c(\beta - b_s)$ for all s.

Note that if α is not rational, the condition $(*)$ is equivalent to each of the following two conditions.

$(*')$ For any left-c.e. approximation $\{a'_s\}_{s\in\omega}$ of α, there exists a left-c.e. approximation $\{b'_s\}_{s\in\omega}$ of β and a constant c such that $\alpha - a'_s \leq c(\beta - b'_s)$ for all s.

$(*'')$ For any left-c.e. approximation $\{b''_s\}_{s\in\omega}$ of β, there exists a left-c.e. approximation $\{a''_s\}_{s\in\omega}$ of α and a constant c such that $\alpha - a''_s \leq c(\beta - b''_s)$ for all s.

These equivalences indicate that the notion of Solovay reducibility is robust and that, intuitively speaking,

$\alpha \leq_S \beta$ amounts to α being approximable by left-c.e. approximations at least as fast as β.

convergence rate of random left-c.e. reals

The degree structure induced by Solovay reducibility on the class of left-c.e. reals has been well studied. One important result is the following theorem by Solovay, and Kučera and Slaman.

Theorem 1 (Solovay; Kučera & Slaman)

A left-c.e. real is Solovay complete if and only if it is Martin-Löf random.

Here, a left-c.e. real is *Solovay complete* if every left-c.e. real is Solovay reducible to it.

This theorem implies that among left-c.e. reals exactly the random ones converge as slowly as possible.

This phenomenon was further explored by [Barmpalias and Lewis-Pye, 2017] and [Miller, 2017].

a derivation on the d.c.e. reals

Fix a random left-c.e. real Ω with a left-c.e. approximation $\{\Omega_s\}_{s\in\omega}$.

Definition 2

Given a d.c.e. real α with a d.c.e. approximation $\{\alpha_s\}_{s\in\omega}$, let

$$\partial\alpha = \partial\{\alpha_s\} = \lim_{s\to\infty} \frac{\alpha - \alpha_s}{\Omega - \Omega_s}.$$

By [Barmpalias and Lewis-Pye, 2017] and [Miller, 2017], $\partial\alpha$ is well-defined and its value is independent of the choice of $\{\alpha_s\}_{s\in\omega}$.

Theorem 3 (Barmpalias & Lewis-Pye; Miller)

$\partial\alpha = 0$ if and only if α is not random.

Corollary 4

All the d.c.e. approximations to a random d.c.e. real converge at exactly the same speed.

Proof.

Let γ be a random d.c.e. real, $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ two d.c.e. approximations of γ. Then $\partial\{a_s\} = \partial\{b_s\} = \partial\gamma \neq 0$, i.e. ,

$$\lim_{s\to\infty} \frac{\gamma - a_s}{\Omega - \Omega_s} = \lim_{s\to\infty} \frac{\gamma - b_s}{\Omega - \Omega_s} \neq 0.$$

Then

$$\lim_{s\to\infty} \frac{\gamma - b_s}{\gamma - a_s} = \lim_{s\to\infty} \frac{\frac{\gamma - b_s}{\Omega - \Omega_s}}{\frac{\gamma - a_s}{\Omega - \Omega_s}} = 1.$$

$\square$

definition of speedability

Instead of comparing the speed of convergence of the approximations of two different left-c.e. reals, one can also consider

- the speed of convergence of different approximations of a single left-c.e. real in general,
- the possibility of speeding up the convergence rate of a single computable approximation.

Definition 5 (speedability for computable approximations)

Let $\{a_s\}_{s\in\omega}$ be a computable approximation of a real α and $\rho \in [0,1)$ be some rational.

- If $a_s = \alpha$ for all but finitely many s, we simply say $\{a_s\}_{s\in\omega}$ is *ρ-speedable*.
- Otherwise, we say $\{a_s\}_{s\in\omega}$ is *ρ-speedable* if there is a nondecreasing computable function $f\colon \mathbb{N} \to \mathbb{N}$ such that

$$\liminf_{s\to\infty} \left| \frac{\alpha - a_{f(s)}}{\alpha - a_s} \right| \leq \rho.$$

We say $\{a_s\}_{s\in\omega}$ is *speedable* if it is ρ-speedable for some $\rho \in (0,1)$.

definition of speedability

Definition 6

Let α be a real number, and ρ be some rational. Let X stand for one of the terms left-c.e., right-c.e., d.c.e., or c.a.

- α is X *(ρ-)speedable* if it has a (ρ-)speedable X approximation.
- α is *strongly* X *(ρ-)speedable* if it has an X approximation that is (ρ-)speedable via the function $s \mapsto s+1$.
- α is *fully* X *speedable* if it is X and all of its X approximations are speedable.
- α is *weakly* X *speedable* if there exist a rational $\rho \in (0,1)$ and two X approximations $\{a_s\}_{s\in\omega}$ and $\{b_s\}_{s\in\omega}$ of α such that

$$\liminf_{s\to\infty} \left| \frac{\alpha - b_s}{\alpha - a_s} \right| \leq \rho. \tag{1}$$

properties of speedability

For all $\mathsf{X} \in \{\text{left-c.e., right-c.e., d.c.e., c.a.}\}$, the following hold.

$$\text{strong } \mathsf{X} \text{ speedability} \Rightarrow \mathsf{X} \text{ speedability} \Rightarrow \text{weak } \mathsf{X} \text{ speedability}$$

$$\text{full } \mathsf{X} \text{ speedability} \Rightarrow \mathsf{X} \text{ speedability}$$

$$\mathsf{X}\ 0\text{-speedability} \Rightarrow \mathsf{X} \text{ speedability}$$

Questions:

- Can any of the above implications be reversed for some X?
- What's the relation between speedability and non-randomness?

left-c.e. speedability

[Merkle & Titov, 2020] studied left-c.e. speedability and proved that for left-c.e. in place of X, most of the implications can be reversed.

Theorem 7 (Merkle & Titov)

For a left-c.e. real γ, the following are equivalent.

1. *γ is left-c.e. speedable.*
2. *γ is fully left-c.e. speedable.*
3. *γ is strongly left-c.e. speedable.*
4. *γ is weakly left-c.e. speedable.*

left-c.e. speedability

By Theorem 3, all left-c.e. random reals are not left-c.e. speedable.

Questions:

Q1: Are all nonrandom left-c.e. reals left-c.e. speedable?

Q2: Are all left-c.e. speedable reals left-c.e. 0-speedable?

- Q1 is answered in the negative by [Hölzel & Janicki, 2023].
- Q2 is answered in the negative by [Hölzel, Janicki, Merkle & Stephan, 2024].

d.c.e. speedability

For d.c.e. in place of X, we show that most of the simple implications can be reversed. Moreover, we prove that a d.c.e. real is d.c.e. speedable if and only if it is nonrandom.

Theorem 8 (Barmpalias, Merkle, Titov, F.)

For a d.c.e. real γ, the following are equivalent.

1. *γ is d.c.e. 0-speedable.*
2. *γ is strongly d.c.e. speedable.*
3. *γ is d.c.e. speedable.*
4. *γ is weakly d.c.e. speedable.*
5. *γ is nonrandom.*

Lemma 9

If γ is a nonrandom d.c.e. real, then γ is strongly d.c.e. 0-speedable.

full d.c.e. speedability

- Due to the negative answer of Q1 by [Hölzel & Janicki, 2023], there is a nonrandom left-c.e. real which is not left-c.e. speedable.
- For this real, it is d.c.e. speedable but not fully d.c.e. speedable, since it has a left-c.e. approximation which is not speedable and all left-c.e. approximations are also d.c.e. approximations.
- Thus, d.c.e. speedability does not imply full d.c.e. speedability.

However, the implication holds when restricting attention to d.c.e. reals that are neither left-c.e. nor right-c.e. .

Theorem 10 (Barmpalias, Merkle, Titov, F.)

If a d.c.e. real is neither left-c.e. nor right-c.e., then it is fully d.c.e. speedable.

Lemma 11

Every two-sided computable approximation is speedable.

c.a. speedability

Corollary 12

If a c.a. real is neither left-c.e. nor right-c.e., then it is fully c.a. speedable.

Proposition 13

Every c.a. real has a two-sided computable approximation.

Corollary 14

Every c.a. real is c.a. speedable.

With a more involved argument, we can show the following stronger result.

Theorem 15

Every c.a. real is strongly c.a. 0-speedable.

a hierarchy of Solovay tests

Definition 16

Let $\{I_i\}_{i\in\omega}$ be a Solovay test where $I_i = [\ell_i, r_i]$. We refer to $\ell_0, r_0, \ell_1, r_1, \ldots$ as the *sequence of endpoints* of this Solovay test, and its *sequence of left endpoints* and *sequence of right endpoints* are defined accordingly. The Solovay test then

- is *converging* if its sequence of endpoints converges,
- is *d.c.e.* if its sequence of endpoints has bounded variation,
- is *left-c.e.* if its sequence of left endpoints is nondecreasing,
- *has bounded increments* if for some rational constant $d > 0$ it holds for all i that

$$\ell_{i+1} \geq \ell_i + d(r_i - \ell_i). \tag{2}$$

a hierarchy of Solovay tests

Proposition 17

(i) *A real is nonrandom and c.a. if and only if it is covered by a converging Solovay test.*

(ii) *A real is nonrandom and d.c.e. if and only if it is covered by a d.c.e. Solovay test.*

(iii) *A real is nonrandom and left-c.e. if and only if it is covered by a left-c.e. Solovay test.*

Proposition 18

A real is left-c.e. speedable if and only if it is covered by a Solovay test that has bounded increments.

speedup functions for approximating Ω

Back to Ω, we failed to characterize left-c.e. random reals by the speedability of its left-c.e. approximations. However, we can still ask the following question.

Question

Which functions speedup Ω, i.e.,

$$\liminf_{s\to\infty} \frac{\Omega - \Omega_{f(s)}}{\Omega - \Omega_s} < 1?$$

Theorem 19

For a strictly increasing function $f : \mathbb{N} \mapsto \mathbb{N}$, if f is dominated by a function g which is computable from some low for Ω set X, then it does not speedup Ω.

Conjecture

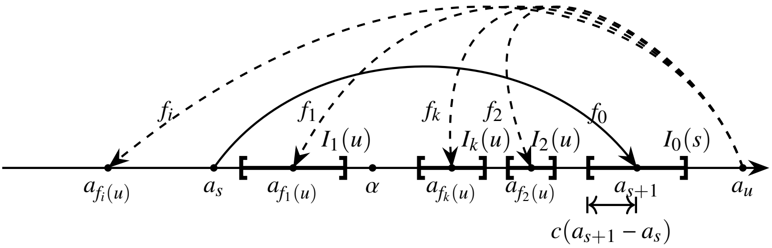
The following are equivalent for a Turing degree $\mathbf{d}$:

- $\mathbf{d}$ does not compute any speedup function for Ω;
- $\mathbf{d}$ is HIF relative to some low for Ω degree.

Proof idea of Lemma 11 (I)

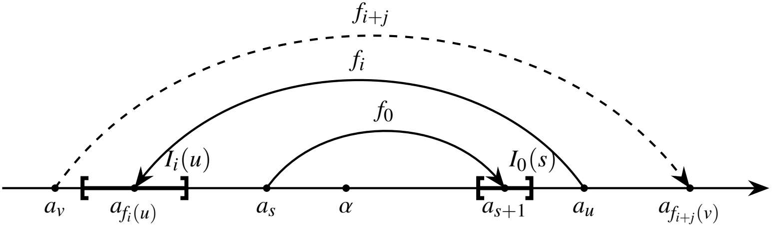
We prove by contradiction. Suppose $\{a_s\}_{s\in\omega}$ is a two-sided computable approximable to real α, and $\{a_s\}_{s\in\omega}$ is not ρ-speedable. Let $s \geq n_{2k}$ be an index such that $a_s < \alpha < a_{s+1}$.

- We inductively define $2k+1$ many computable functions $f_i\colon \mathbb{N} \to \mathbb{N}$ for $0 \leq i \leq 2k$.
- We first find the index $u \geq s+1$ such that a_u is the last one to the right of or equal to a_{s+1}.
- Then all of $a_{f_1(u)}, a_{f_2(u)}, \ldots, a_{f_k(u)}$ are to the left of a_{s+1}.
- Moreover, they must jump over the interval $[a_{s+1} - c(a_{s+1} - a_s), a_{s+1}]$. Then their distances with a_u is bounded below.
- Thus, by our arrangement of the parameters, we argue that at least one of them must be to the left of a_s.



Proof idea of Lemma 11 (II)

- Fixing one such point $a_{f_i(u)}$, we find the index $v \geq f_i(u)$ such that a_v is the last one to the left of or equal to $a_{f_i(u)}$.
- In a symmetrical way as before, we find one point $a_{f_{i+j}(v)}$ to be right of a_u.
- However this contradicts to the fact that a_u is the last one to the right of or equal to a_{s+1}.



Thank you for your attention!